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Cycles over Fields of Transcendence Degree 1

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Introduction

We work over subfields k of $\mathbb{C}$, the field of complex numbers. For a smooth variety V over k, the Chow group of cycles of codimension p is defined (see [F]) as

$$\mathrm{CH}^p(V) = \frac{Z^p(V)}{R^p(V)},$$

where (a) the group of cycles $Z^p(V)$ is the free abelian group on scheme-theoretic points of V of codimension p and (b) rational equivalence $R^p(V)$ is the subgroup generated by cycles of the form $\mathrm{div}_W(f)$, where W is a subvariety of V of codimension $p-1$ and f is a nonzero rational function on it. There is a natural cycle class map

$$\mathrm{cl}_p\colon \mathrm{CH}^p(V) \to \mathrm{H}^{2p}(V),$$

where the latter denotes the singular cohomology group $\mathrm{H}^{2p}(V(\mathbb{C}), \mathbb{Z})$ with the (mixed) Hodge structure given by Deligne (see [D]). The kernel of cl_p is denoted by $F^1\,\mathrm{CH}^p(V)$. There is an Abel–Jacobi map (see [G])

$$\Phi_p\colon F^1\,\mathrm{CH}^p(V) \to \mathrm{IJ}^p(\mathrm{H}^{2p-1}(V)),$$

where the latter is the intermediate Jacobian of a Hodge structure and defined as

$$\mathrm{IJ}^p(H) = \frac{H \otimes \mathbb{C}}{F^p(H \otimes \mathbb{C}) + H}.$$

The kernel of Φ_p is denoted by $F^2\,\mathrm{CH}^p(V)$.

Conjecture 1 (Bloch–Beilinson). *If V is a variety defined over a number field k, then $F^2\,\mathrm{CH}^p(V) = 0$.*

We (of course) offer no proof of this conjecture. However, there are examples due to Schoen and Nori (see [S]) showing that one cannot relax the conditions in this conjecture. In this paper we present these and other examples to show that $F^2\,\mathrm{CH}^2(V)$ is nonzero for V a variety over a field of transcendence degree ≥ 1 whenever it is nonzero over some larger algebraically closed field.

We begin in Section 1 with a lemma. We then apply this lemma to prove the following result (using Hodge-theoretic methods) for the second symmetric power of a curve.

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