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Plurisubharmonic Lyapunov Functions

Małgorzata Stawiska

Stanisław Łojasiewicz in memoriam

1. Introduction

In the study of dynamics of a continuous map $f: X \mapsto X$ on a compact metric space X, one is often interested in f-invariant sets or measures. When $f: \mathbb{CP}^k \mapsto \mathbb{CP}^k$ is a holomorphic endomorphism of degree $d \geq 2$, such invariant objects can be constructed by means of the function

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log\|F^n(z)\|, \quad z \in \mathbb{C}^{k+1}$$

(cf. [HP; Ue]), where F is a lift of f to $\mathbb{C}^{k+1}$, that is, $\pi \circ F = f \circ \pi$ with π: $\mathbb{C}^{k+1} \setminus \{0\} \mapsto \mathbb{CP}^k$ the standard projection map. Each coordinate of F is a homogeneous polynomial of degree d and $F^{-1}(0) = 0$. It is easy to see that G is a plurisubharmonic (PSH) function on $\mathbb{C}^{k+1}$ that is not identically equal to $-\infty$, is continuous on $\mathbb{C}^{k+1} \setminus \{0\}$, and satisfies $G(F(z)) = d \cdot G(z)$ for $z \in \mathbb{C}^{k+1}$. Using G, one defines a positive closed (1, 1)-current T by $\pi^*T = dd^cG$, and subsequently $T^l = T \wedge \cdots \wedge T$ $(l = 2, \dots, k)$. Note that $\mu = T^k$ is a Borel finite measure on $\mathbb{CP}^k$. These currents and their supports satisfy the invariance conditions $f^*(T^l) = d^l \cdot T^l$ and $f^{-1}(\operatorname{supp} T^l) = \operatorname{supp} T^l = f(\operatorname{supp} T^l)$ for $l = 1, \dots, k$.

The function G has other properties of interest from the dynamical systems point of view. Note that 0 is an attracting fixed point for F. It was proven in [Ue] and [HP] that the basin of attraction $\mathcal{A}$ of 0, defined as $\mathcal{A} = \{z \in \mathbb{C}^{k+1} : F^n(0) \to 0$ as $z \to 0\}$, equals $\{z \in \mathbb{C}^{k+1} : G(z) < 0\}$. Also, $\mathcal{A}$ is a bounded domain. The equation $G \circ F = d \cdot G$ implies that in $\mathcal{A}$, $-G$ increases along the orbits of F (i.e., it is a Lyapunov function for F). Although G is commonly referred to as the "dynamical Green function", it seems that no proof has been given that it is indeed a Green function in any sense used in complex analysis. In fact, G is the pluricomplex Green function of $\mathcal{A}$ with logarithmic pole at the point 0 (see Proposition 3).

If the restriction of the holomorphic map $f: \mathbb{CP}^k \mapsto \mathbb{CP}^k$ to $\mathbb{C}^k \cong [z_1 : z_2 : \cdots : 1]$ is a regular polynomial endomorphism of $\mathbb{C}^k$ (i.e., if $f|_{\mathbb{C}^k} = (f_1, \dots, f_k): \mathbb{C}^k \mapsto \mathbb{C}^k$ is a polynomial map with $\deg f_j = d$, $j = 1, \dots, k$, such that the homogeneous parts of f_j of degree d have a common zero only at the origin), then one obtains a continuous plurisubharmonic function by taking

$$g(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+\|f^n(z)\|, \quad z = (z_1, \dots, z_k) \in \mathbb{C}^k.$$

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