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Tartar Conjecture and Beltrami Operators

Daniel Faraco

1. Introduction

The study of the oscillation of sequences of gradients of Sobolev functions is a central topic in the calculus of variations and nonlinear partial differential equations. In this note we examine which subsets of the space of 2×2 matrices $\mathbf{M}^{2\times 2}$ "forbid" the oscillation of sequences of gradients approaching them. In precise words: Let E be a subset of $\mathbf{M}^{2\times 2}$ and let $1 \le p \le \infty$. Is the set stable in the sense that every sequence $\{f_j\}$ such that

$$\operatorname{dist}_E(Df_j) \to 0 \text{ in } L^p \tag{1.1}$$

satisfies that $\{Df_j\}$ is compact in L^p? The natural tool for studying this problem is Young measures, which have proved to be an efficient device for capturing properties of oscillating sequences. In particular, we will need the following type of Young measures.

Definition 1.1. Let $1 \le p \le \infty$. A probability measure $\nu \in \mathcal{M}(\mathbf{M}^{2\times 2})$ is a $W^{1,p}$ *homogeneous gradient Young measure* if there exists a sequence $\{f_j\}$, weakly convergent in $W^{1,p}(\Omega, \mathbb{R}^2)$, such that for every $\varphi \in C_0^\infty(\mathbf{M}^{2\times 2})$ we have

$$\varphi(Df_j) \stackrel{\star}{\rightharpoonup} \int_{\mathbf{M}^{2\times 2}} \varphi(\lambda)\, d\nu(\lambda) \text{ in } L^\infty(\Omega).$$

The set of $W^{1,p}$ homogeneous gradient Young measures supported in a closed set E is denoted by $\mathcal{H}^p(E)$.

The sequence $\{Df_j\}$ is called the *generating* sequence. A standard covering argument shows that Definition 1.1 does not depend on the domain Ω. It follows from [P, Prop. 6.12] that, if the sequence $\{|Df_j|^p\}$ is known to be weakly convergent in $L^1(\Omega)$, then we are concerned with the following question.

Question 1.2. Which closed sets $E \in \mathbf{M}^{2\times 2}$ satisfy

$$\mathcal{H}^p(E) = \{\delta_A : A \in E\}? \tag{1.2}$$

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