Michigan Math. J. 52 (2004)

A Purity Theorem for Abelian Schemes

Adrian Vasiu

1. Introduction

Let K be the field of fractions of a discrete valuation ring O. Let Y be a flat O-scheme that is regular, and let U be an open subscheme of Y whose complement in Y is of codimension in Y at least 2. We call the pair (Y, U) an extensible pair. Let $q : \mathcal{S} \to \mathrm{Sch}_O$ be a stack over the category Sch_O of O-schemes endowed with the Zariski topology. Let $\mathcal{S}_Z$ be the fibre of q over an O-scheme Z. Answers to the following question provide information on $\mathcal{S}$.

Question 1.1. Is the pull-back functor $\mathcal{S}_Y \to \mathcal{S}_U$ surjective on objects?

Question 1.1 has a positive answer in any one of the following three cases:
(i) $\mathcal{S}$ is the stack of morphisms into the Nèron model over O of an abelian variety over K, and Y is smooth over O (see [N]);
(ii) $\mathcal{S}$ is the stack of smooth, geometrically connected, projective curves of genus at least 2 (see [M-B]);
(iii) $\mathcal{S}$ is the stack of stable curves of locally constant type, and there is a divisor DIV of Y with normal crossings such that the reduced scheme $Y \setminus U$ is a closed subscheme of DIV (see [dJO]).

Let p be a prime. If the field K is of characteristic 0, then an example of Raynaud–Gabber–Ogus shows that Question 1.1 does not always have a positive answer if $\mathcal{S}$ is the stack of abelian schemes (see [dJO, Sec. 6]). This invalidates [FaC, Chap. IV, Thms. 6.4, 6.4′, 6.8] and leads to the following problem.

Problem 1.2. Classify all those Y with the property that, for any extensible pair (Y, U) with U containing Y_K, every abelian scheme (resp., every p-divisible group) over U extends to an abelian scheme (resp., to a p-divisible group) over Y.

We call such Y a healthy (resp., p-healthy) regular scheme (cf. [V, 3.2.1(2), (9)]. The counterexample of [FaC, p. 192] and the classical purity theorem of [G, p. 275] indicate that Problem 1.2 is of interest only if K is of characteristic 0 (resp., only if O is a faithfully flat $\mathbb{Z}_{(p)}$-algebra). We shall therefore assume hereafter that O is of mixed characteristic $(0, p)$. Let $e \in \mathbb{N}$ be the index of ramification of O. If $e \leq p - 2$, then a result of Faltings states that Y is healthy and p-healthy regular,

Received September 16, 2002. Revision received August 15, 2003.