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A Heat Kernel Lower Bound for Integral Ricci Curvature

Xianzhe Dai & Guofang Wei

1. Introduction

The heat kernel is one of the most fundamental quantities in geometry. It can be estimated both from above and below in terms of Ricci curvature (see [1; 2; 7]). The heat kernel upper bound has been extended to integral Ricci curvature by Gallot in [4]. Here we extend Cheeger and Yau's [2] lower bound to integral Ricci curvature.

Our notation for the integral curvature bounds on a Riemannian manifold (M, g) is as follows. For each $x \in M$ let $r(x)$ denote the smallest eigenvalue for the Ricci tensor $\mathrm{Ric}\colon T_xM \to T_xM$, and for any fixed number λ define

$$\rho(x) = |\min\{0, r(x) - (n-1)\lambda\}|.$$

Then set

$$k(p,\lambda,R) = \sup_{x\in M}\left(\int_{B(x,R)} \rho^p\right)^{1/p},$$

$$\bar{k}(p,\lambda,R) = \sup_{x\in M}\left(\frac{1}{\operatorname{vol} B(x,R)}\cdot\int_{B(x,R)} \rho^p\right)^{1/p}.$$

These curvature quantities evidently measure how much Ricci curvature lies below $(n-1)\lambda$ in the (normalized) integral sense. Observe that $\bar{k}(p,\lambda,R) = 0$ if and only if $\mathrm{Ric} \ge (n-1)\lambda$.

Let $E(x,y,t)$ denote the heat kernel of the Laplace–Beltrami operator on a closed manifold (M, g). For any real number λ, we use $E_\lambda(\overline{x,y},t)$ to denote the heat kernel on the model space of constant curvature λ. Our main result is as follows.

Theorem 1.1. *Let $n > 0$ be an integer, let $p > n/2$ and $\lambda \le 0$ be real numbers, and let $D > 0$. Then there exists an explicitly computable $\varepsilon_0 = \varepsilon(n,p,\lambda,D)$ such that, for any (M,g) with $\operatorname{diam} M \le D$ and for $\bar{k}(p,\lambda,D) \le \varepsilon_0$ and $k(p,\lambda,R) \le 1$,*

$$E(x,y,t) \ge E_\lambda(\overline{x,y},t) - (k(p,\lambda,D))^{1/2} C(n,p,\lambda,D)(t^{-(n+1)/2} + 1)$$

for any $x, y \in M$ and $t > 0$.

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