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Meromorphic Vector Fields and Elliptic Fibrations

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1. Introduction

The purpose of this paper is to present a classification of meromorphic semi-complete vector fields under a mild additional assumption. Motivation for this result comes in part from the fact that methods of differential equations and singularity theory can be employed to study certain problems in complex algebraic geometry. Indeed, these problems can be thought of as special cases of more general questions involving differential equations. Here we address some of these questions about differential equations. We also indicate how part of the theory of elliptic surfaces developed by Kodaira can be considered as a particular case of our methods. It should be noted that the mild assumption mentioned previously is always verified in these applications.

Another motivation for our classification is that it generalizes previous works on holomorphic semi-complete vector fields whose interest was already settled. An important reason for considering this generalization is the fact that meromorphic vector fields are more flexible than holomorphic ones in the sense that they appear in several situations where the corresponding holomorphic vector fields do not exist (a simple example being elliptic surfaces, cf. Example 2 to follow). A general classification of meromorphic semi-complete vector fields is interesting and would have additional applications (see Example 1); in fact, it would also be a rather significant generalization of certain natural questions in complex geometry. Whereas the discussion here combined with the results obtained in previous papers about *holomorphic* vector fields may lead to such classification, this attempt would take us too far from the aim of the present article.

We say that a singular holomorphic foliation $\mathcal{F}$ defined on a *neighborhood* of $(0, 0) \in \mathbb{C}^2$ has infinitely many leaves accumulating on $(0, 0)$ if, for a small ball $B(\varepsilon)$ centered at $(0, 0)$, the singular foliation $\mathcal{F}|_B$ of $B(\varepsilon)$ obtained by restriction of $\mathcal{F}$ to $B(\varepsilon)$ possesses infinitely many leaves accumulating on $(0, 0)$. The main result of this paper is the following theorem.

Main Theorem. *Let Y be a holomorphic vector field with an isolated singularity at $(0, 0) \in \mathbb{C}^2$ that has only a finite number of orbits accumulating on the origin. Consider a meromorphic (nonholomorphic) function f defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$. Assume that $X = fY$ is a meromorphic semi-complete*

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