Michigan Math. J. 51 (2003)

The Algebra of K-Invariant Vector Fields on a Symmetric Space G/K

Ilka Agricola & Roe Goodman

1. Introduction

In this paper we study the infinite-dimensional Lie algebra of K-invariant vector fields on a reductive symmetric space G/K. Our motivation was the investigation of the algebra of invariant differential operators for nontransitive group actions on smooth affine varieties and, in particular, the abstract Howe duality theorem one has for this situation (see e.g. [A2, Satz 2.2]). Correspondingly, we shall work in the algebraic category—that is, where G is a complex connected reductive linear algebraic group and K consists of the fixed points of an involutory automorphism θ of G (thus G/K is the complexification of a Riemannian symmetric space).

There is a canonical G-module isomorphism between the space $\mathfrak{X}(G/K)$ of regular algebraic vector fields on G/K and the algebraically induced representation $\mathrm{Ind}_K^G(\sigma)$, where σ is the isotropy representation of K. In particular, the space $\mathfrak{X}(G/K)^K$ of K-invariant vector fields on G/K corresponds to the K-fixed vectors in the induced representation. When G is simple and simply connected, Richardson's results [Ri2] imply that $\mathfrak{X}(G/K)$ is a free module over the algebra $\mathcal{J}$ of K-biinvariant functions on G. In Theorem 2.2 we obtain an explicit set of free generators for a localization $\mathfrak{X}(G/K)_\psi^K$ for some $\psi \in \mathcal{J}$.

We next study $\mathfrak{X}(G/K)^K$ as a Lie algebra and, in Section 3, obtain a formula for the commutator of K-invariant vector fields in terms of the associated K-covariant mappings. The Cartan embedding $G/K \to P \subset G$ given by $gK \mapsto g\theta(g)^{-1}$ is a fundamental tool in the study of symmetric spaces, and it is natural to use it to study $\mathfrak{X}(G/K)^K$. Invariant vector fields on G/K whose horizontal lifts to G are tangent to P are called *flat* (in fact, the Cartan embedding induces *a priori* two different notions of flatness, which we show to be equivalent). We obtain a commutator formula with no curvature term for the action on P of these vector fields. For G simple and simply connected, we prove (Theorem 3.1) that every element of $\mathfrak{X}(G/K)^K$ is flat if and only if K is semisimple (i.e., G/K is not the complexification of a hermitian symmetric space).

In Section 4 we study the conjugation action of a semisimple group G on itself. This is an example of the Cartan embedding of a symmetric space for the group $G \times G$ and involution $\theta(g, h) = (h, g)$. In this case, all conjugation-invariant

Received July 17, 2002. Revision received March 28, 2003.

This work was partially supported by the SFB 288 “Differential geometry and quantum physics” of the Deutsche Forschungsgemeinschaft.