Michigan Math. J. 51 (2003)

# Finiteness Results for Multiplicatively Dependent Points on Complex Curves

E. Bombieri, D. W. Masser, & U. Zannier

## 1. Introduction

Let $n$ be a positive integer, and let $C$ be a curve in $\mathbf{G}_m^n$ that we suppose (for convenience) is absolutely irreducible. When $H$ is a fixed algebraic subgroup of $\mathbf{G}_m^n$, the intersection of $C$ with $H$ by itself is relatively easy to determine. In [BMZ] we began to study in this context the union of all algebraic subgroups restricted only by dimension. In particular: if $n \geq 2$, and if $C$ is defined over the field $\bar{\mathbf{Q}}$ of all algebraic numbers and satisfies a fairly natural extra hypothesis, then it was shown (Thm. 2, p. 1121) that the intersection of $C$ with the union $\mathcal{H}_{n-2}$ of all $H$ of dimension at most $n-2$ is a finite (possibly empty) set.

The main purpose of the present paper is to generalize this result with regard to the field of definition. More precisely, we shall prove the following.

Theorem. *Let $K$ be a field of characteristic zero, and for $n \geq 2$ let $C$ be an irreducible curve in $\mathbf{G}_m^n$ that is defined over the algebraic closure $\bar{K}$ and is not contained in any translate of an algebraic subgroup of dimension at most $n-1$. Then the intersection of $C$ with the union $\mathcal{H}_{n-2}$ of all algebraic subgroups of dimension at most $n-2$ is a finite* (*possibly empty*) *set.*

The restriction to characteristic zero is necessary here. For example, if $C$ is any curve defined over the finite field $K = \mathbf{F}_p$, then the set $C(\bar{K})$ is infinite; on the other hand, any nonzero element of $\bar{K} = \bar{\mathbf{F}}_p$ is a root of unity and so $C(\bar{K})$ lies in the union $\mathcal{H}_0$ of all algebraic subgroups of dimension 0.

We recover Theorem 2 of [BMZ] by taking $K$ as the field $\mathbf{Q}$ of all rational numbers in our Theorem. Our generalization enables us to deduce consequences, however, over the complex field $\mathbf{C}$; thus, if $z_1, \dots, z_n$ are any distinct complex numbers then we see that there are only finitely many complex numbers $z \neq z_1, \dots, z_n$ for which there are two relations,

$$(z - z_1)^{a_1} \cdots (z - z_n)^{a_n} = (z - z_1)^{b_1} \cdots (z - z_n)^{b_n} = 1,$$

where $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ in $\mathbf{Z}^n$ are linearly independent over $\mathbf{Q}$. This follows immediately from our Theorem by considering the line $C$ parameterized by $z - z_1, \dots, z - z_n$ as $z$ varies.

Received January 3, 2002. Revision received June 25, 2003.