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Second Variation of Compact Minimal Legendrian Submanifolds of the Sphere

Francisco Urbano

1. Introduction

The second variation operator of minimal submanifolds of Riemannian manifolds (the *Jacobi operator*) carries information about stability properties of the submanifold when it is thought of as a critical point for the area functional. When the ambient Riemannian manifold is a sphere $\mathbb{S}^m$, Simons [S] characterized the totally geodesic submanifolds as the minimal submanifolds of $\mathbb{S}^m$ either with the lowest index (number of independent infinitesimal deformations that do decrease the area) or with lowest nullity (dimension of the Jacobi fields, i.e., infinitesimal deformations through minimal immersions). Other results about the index and the nullity of minimal surfaces of the sphere can be found in [E2; MU; U1]. If m is odd (i.e., if $m = 2n+1$) then one can consider n-dimensional minimal *Legendrian* submanifolds of $\mathbb{S}^{2n+1}$ (see Section 2 for the definition). These submanifolds are particulary interesting because the cones over them are special Lagrangian submanifolds of the complex Euclidean space $\mathbb{C}^{n+1}$, and as Joyce pointed out in [J, Sec. 10.2], the knowledge of their index is deeply related to the dimension of the moduli space of asymptotically conical special Lagrangian submanifolds of $\mathbb{C}^{n+1}$. This fact, joint to the characterization of minimal Legendrian submanifolds given by Lê and Wang in [LW], directed my attention to the study of the second variation of minimal Legendrian submanifolds of odd-dimensional spheres.

In Section 2 we compute the Jacobi operator of compact minimal Legendrian submanifolds of $\mathbb{S}^{2n+1}$, proving that it is an intrinsic operator on the submanifold and that it can be written in terms of the exterior differential, its codifferential operator, and the Laplacian (see formula (2)). In Section 3 we decompose the Jacobi operator as the sum of two elliptic operators and then study their indexes and nullities (Theorem 1 and Corollary 1). As a consequence we obtain a formula for the index and the nullity of compact minimal Legendrian submanifolds of $\mathbb{S}^{2n+1}$ (Corollary 2). Finally, we particularize our study to compact minimal Legendrian surfaces of $\mathbb{S}^5$ and prove the following result.

If M is an orientable compact minimal (*non–totally geodesic*) *Legendrian surface in* $\mathbb{S}^5$, *then*:

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