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The Third Cauchy–Fantappiè Formula of Leray

Cezar Joiţa & Finnur Lárusson

Introduction

In the third paper [L1] in his famous series *Problème de Cauchy,* Jean Leray founded the modern theory of residues. This paper is reprinted in Leray's *Œuvres scientifiques* [L2] with an introduction by G. M. Henkin. In it, almost as an aside, Leray presents three representation formulas for holomorphic functions on a domain in affine space, calling them the first, second, and third Cauchy–Fantappiè formulas. The first formula, nowadays often called the Cauchy–Leray formula, is truly fundamental: most integral representation formulas can be derived from it in one way or another. For an introduction to this area of complex analysis, see [B] and [K]. The second and third formulas, obtained from the first one using residue theory, have received little attention in the literature. The proof of the second formula is straightforward, but the third one turns out to be quite subtle. We will show by means of examples that it does not hold without some additional conditions not mentioned by Leray. We present and study sufficient conditions and a necessary condition for the third formula to hold and, in the case of a contractible domain, characterize it cohomologically.

We assume that the reader is familiar with approximately the first half of Leray's paper [L1], including the coboundary map and the associated long exact homology sequence, absolute and relative residues, the residue formula, and the interaction of the residue map with several natural cohomology maps. To establish a context and notation, we begin by reviewing Leray's derivation of the three formulas.

1. The Cauchy–Fantappiè Formulas

We start with a number of definitions, following Leray. Let X be a domain in $\mathbb{C}^n$ $(n \geq 1)$, and let $Y = \mathbb{P}^n \times X$. Leray assumes that X is convex, but we do not. We define

$$Q = \{(\xi, x) \in Y : \xi \cdot x = \xi_0 + \xi_1 x_1 + \cdots + \xi_n x_n = 0\}$$

and

$$P_z = \{(\xi, x) \in Y : \xi \cdot z = 0\} = \text{hyperplane} \times X$$

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