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On the Geography of Stein Fillings of Certain 3-Manifolds

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1. Introduction

Let (Y, ξ) be a given (closed) contact 3-manifold. (For basic definitions regarding contact structures the reader is advised to consult e.g. [Ae; Et2].) A 4-manifold X is called a *Stein filling* of (Y, ξ) if X is a sublevel set of a plurisubharmonic function on a Stein surface and if (Y, ξ) is contactomorphic to ∂X (the contact structure induced by the complex tangencies). For a more detailed account regarding Stein fillings, see [LiMa].

Inspired by the geography problem of minimal surfaces of general type—that is, the determination of pairs (signature, Euler characteristic) $= (\sigma, \chi)$ of such 4-manifolds—we are led to the following.

Problem 1.1. Describe characteristic numbers of Stein fillings of a given contact 3-manifold (Y, ξ).

This problem has been solved for particular 3-manifolds—such as the 3-sphere S^3, the Poincaré sphere $\Sigma(2, 3, 5)$ (with both orientations), and lens spaces with specific contact structures—in a much stronger sense: for these examples also, the diffeomorphism classification of Stein fillings has been achieved (see [E1; Mc; OO]). For related results concerning $-\Sigma(2, 3, 11)$ and the 3-torus T^3, see [St1]. These examples led us to the following conjecture.

Conjecture 1.2. *The set*

$$\mathcal{CF}_{(Y,\xi)} = \{b_1(W), \sigma(W), \chi(W) \mid W \text{ is a Stein filling of } (Y, \xi)\}$$

of the characteristic numbers of Stein fillings of a given 3*-manifold* (Y, ξ) *is a finite set*.

Remark 1.3. A related conjecture could be formulated by examining finiteness properties of the set of diffeomorphism types of Stein fillings of a given contact 3-manifold (Y, ξ). In the light of a recent observation of I. Smith, this conjecture is too ambitious in general.

Our main result in this paper makes a minor step for verifying Conjecture 1.2 in general—and, in fact, proves the conjecture in some particular cases; see Corollaries 1.5 and 1.7.

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