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Constructions of Nontautological Classes on Moduli Spaces of Curves

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0. Introduction

The *tautological rings* $R^*(\bar{M}_{g,n})$ are natural subrings of the Chow rings of the Deligne–Mumford moduli spaces of pointed curves:

$$R^*(\bar{M}_{g,n}) \subset A^*(\bar{M}_{g,n}) \tag{1}$$

(the Chow rings are taken with $\mathbb{Q}$-coefficients). The system of tautological subrings (1) is defined to be the set of smallest $\mathbb{Q}$-subalgebras satisfying the following three properties [FP].

(i) $R^*(\bar{M}_{g,n})$ contains the cotangent line classes

$$\psi_1, \dots, \psi_n \in A^1(\bar{M}_{g,n}).$$

(ii) The system is closed under push-forward via all maps forgetting markings:

$$\pi_*\colon R^*(\bar{M}_{g,n}) \to R^*(\bar{M}_{g,n-1}).$$

(iii) The system is closed under push-forward via all gluing maps:

$$\pi_*\colon R^*(\bar{M}_{g_1,n_1\cup\{*\}}) \otimes_{\mathbb{Q}} R^*(\bar{M}_{g_2,n_2\cup\{\bullet\}}) \to R^*(\bar{M}_{g_1+g_2,n_1+n_2}),$$

$$\pi_*\colon R^*(\bar{M}_{g_1,n_1\cup\{*,\bullet\}}) \to R^*(\bar{M}_{g_1+1,n_1}).$$

The tautological rings possess remarkable algebraic and combinatorial structures with basic connections to topological gravity. A discussion of these properties together with a conjectural framework for the study of $R^*(\bar{M}_{g,n})$ can be found in [F; FP].

In genus 0, the equality

$$R^*(\bar{M}_{0,n}) = A^*(\bar{M}_{0,n})$$

for $n \geq 3$ is well known from Keel's study [K].

Denote the image of $R^*(\bar{M}_{g,n})$ under the canonical map to the ring of *even* cohomology classes by

$$RH^*(\bar{M}_{g,n}) \subset H^{2*}(\bar{M}_{g,n}).$$

In genus 1, Getzler has claimed the isomorphisms

$$R^*(\bar{M}_{1,n}) \cong RH^*(\bar{M}_{1,n})$$

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