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Some Refined Schwarz–Pick Lemmas

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1. Introduction

There has recently been an unexpected activity in the discovery of some new and sharper versions of the Schwarz lemma (see [1; 2; 3; 7; 8; 9]), mostly in the disc. It is our aim in this paper to obtain further forms in the spirit of Pick for functions between discs and between balls of different dimensions.

Several new forms in the disc [1; 7; 9] consider functions f that map the origin to the origin and then use the functions $f(z)/z$ to refine the estimates given in the classical forms of the Schwarz lemma. We first relax the condition $f(0) = 0$ and obtain the Pick versions of these forms. The other generalization comes from replacing the function $f(z)/z$ by a suitable variant for functions between balls. The results in these directions are given in two sections: those pertaining to derivatives in Section 4; and those about the function values in Section 5. In Section 6, we apply the results of Section 4 to obtain quantitative estimates on derivatives at boundary points—in particular, on angular derivatives in discs and in balls. Two earlier sections are for the preliminary material: Section 2 for the more analytic; and Section 3 for the more geometric.

Our notation is fully explained in Section 2, but we mention two points in order to give a sample of our main results. First, β_n denotes the integrated Bergman metric in the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. Second, $f^*(a) = \varphi'_{f(a)}(f(a))f'(a)\varphi'_a(0)$ is the hyperbolic derivative of a holomorphic function f at a point $a \in \mathbb{B}_n$, where φ_a is a Möbius transformation that interchanges 0 and a. In this work, the ball case always includes the disc case.

Lemma 4.1. *If $f : \mathbb{B}_n \to \mathbb{B}_m$ is holomorphic, $f(a) = b$, and $f(A) = B$, then*

$$\beta_m\left(Sf^*(a)\frac{\varphi_a(A)}{|\varphi_a(A)|}, Tf^*(A)\frac{\varphi_A(a)}{|\varphi_A(a)|}\right) \le 2\beta_n(a, A),$$

where S and T are unitary transformations satisfying $S(\varphi_b(B)) = T(\varphi_B(b))$.

Corollary 5.3. *If $\zeta \in \mathbb{B}_n$ then, for all holomorphic $f : \mathbb{B}_n \to \mathbb{B}_m$ satisfying $f(a) = b$ and $f(A) = B$, the region of values of $f^*(A)\zeta$ is a closed ellipsoid whose center is*

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