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Foliation by Graphs of CR Mappings and a Nonlinear Riemann–Hilbert Problem for Smoothly Bounded Domains

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1. Introduction

Let D be a bounded, smoothly bounded domain in $\mathbb{C}^\ell$, $\ell \geq 2$, and let M be a real C^∞ submanifold of $\partial D \times \mathbb{C}^m$, $m \geq 1$. We here address the question of when there exists a mapping f such that

(RH) $f : \bar{D} \to \mathbb{C}^m$ is continuous on $\bar{D}$ and analytic on D such that the graph of f over ∂D is contained in M.

A problem where one is required to find an f satisfying (RH) is often called a *Riemann–Hilbert* problem; Riemann proposed such a question for $\ell = m = 1$ in 1851. We shall refer to the problem of finding an f satisfying (RH) as the Riemann–Hilbert problem for M. If an f exists satisfying (RH) then we shall say that the Riemann–Hilbert problem for M is *solvable* and that f is a *solution*. For $z \in \partial D$, let $M_z \equiv \{w \in \mathbb{C}^m : (z, w) \in M\}$. We will say here that the Riemann–Hilbert problem (RH) is *linear* if, for every $z \in \partial D$, M_z is a real affine subspace of $\mathbb{C}^m$. We shall say that the Riemann–Hilbert problem (RH) is *nonlinear* if it is not linear.

For the case $\ell = 1$ we mention the references [B1; B2; Fo; HMa; S; Sh1; Sh2; V; We1–We5]. See [We5] for a useful survey and reference list. For $\ell \geq 2$, see [B2; B3; BD; D1; D2].

We will first address the following more general question. Let S be a C^∞ CR manifold in $\mathbb{C}^\ell$ (e.g., a real hypersurface in $\mathbb{C}^\ell$) and let M be a real C^∞ submanifold of $S \times \mathbb{C}^m$, $m \geq 1$. Let $z^0 \in S$ and U a neighborhood of z^0 in S. Does there exist a CR map $f : U \to \mathbb{C}^m$ whose graph in $U \times \mathbb{C}^m$ is contained in M? We shall establish conditions under which the answer to this question is "yes". For the case where S is a complex manifold, this question is addressed by theorems in [F1–F3; Kr; So1; So2]. We consider the case of more general S. Applying our result to the case where S is the boundary of an open set in $\mathbb{C}^\ell$ with C^∞ boundary (so S is a real hypersurface in $\mathbb{C}^\ell$), we shall establish conditions where the Riemann–Hilbert problem for M is solvable.

For the more general question of the previous paragraph, assumptions we shall make will imply that the set M is a CR manifold. Under conditions outlined in

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