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A Gluing Formula for the Seiberg–Witten Invariant along T^3

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1. Introduction

This paper is a continuation of studies initiated in [P1]. For the definition and basic properties of the Seiberg–Witten monopole invariant, we refer the reader to the bibliography in [P1]. Our hope is that these studies will ultimately yield a useful theory of Floer-type cohomology for 3-manifolds that is *infinitely* generated. The present goal of this paper is to provide a method of computing the Seiberg–Witten (SW) invariant of a smooth 4-manifold that can be decomposed into two parts along an embedded 3-torus. Under some mild assumptions, we prove a gluing formula for the SW invariant in terms of products of suitably perturbed relative SW invariants of the two end pieces whose common boundary is T^3. In particular, our formula does not require that one of the glued-up pieces be $T^2 \times D^2$, as is the case in [MMS]. We shall derive some interesting applications of this product formula and others in future work [P2].

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2. Perturbed Solutions over the 3-Torus

We study the Seiberg–Witten equations over the 3-manifold $Y = T^3$. We shall always view Y as the trivial S^1 bundle over the 2-torus. Let Σ be the base space T^2; that is, $Y = \Sigma \times S^1$. Note that Y is the unit circle bundle of the canonical line bundle K_Σ over Σ $(\deg(K_\Sigma) = 0)$.

Choose a constant curvature connection on the unit circle bundle Y and let $i\zeta$ denote the corresponding connection form. Let g_Σ be a constant curvature metric on the surface Σ, normalized so that the area of Σ is equal to 1. We endow Y with the metric

$$h_Y = \zeta \otimes \zeta + \pi^*(g_\Sigma),$$

where $\pi : Y \to \Sigma$ is the bundle projection map. Of course, the global 1-form ζ allows a reduction in the structure group of TY to SO(2), and the Levi–Civita

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