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Bordism of Unoriented Surfaces in 4-Space

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1. Introduction

Sanderson [9; 10] studied the group $L_{m,n}$ of bordism classes of "oriented" closed $(m-2)$-manifolds of n components in $\mathbf{R}^m$. He showed that $L_{m,n}$ is isomorphic to the homotopy group $\pi_m\big(\bigvee_{i=1}^{n-1} S^2\big)$; in particular, the bordism group $L_{m,n}$ for $m = 4$ is given as follows.

Theorem 1.1 (Sanderson).

$$L_{4,n} \cong (\underbrace{\mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2}_{\frac{n(n-1)}{2}}) \oplus (\underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{\frac{n(n-1)(n-2)}{3}}).$$

In particular, we have $L_{4,1} \cong \{0\}$, $L_{4,2} \cong \mathbf{Z}_2$, *and* $L_{4,3} \cong \mathbf{Z}_2^3 \oplus \mathbf{Z}^2$.

Similarly, there is a group of bordism classes of "unoriented" closed $(m-2)$-manifolds of n components in $\mathbf{R}^m$. We denote the group by $UL_{m,n}$. The aim of this paper is to determine the bordism group $UL_{m,n}$ for $m = 4$ via purely geometric techniques.

An *n-component surface link F* is a closed surface embedded in $\mathbf{R}^4$ (smoothly, or PL and locally flatly) such that an integer in $\{1, \dots, n\}$, called the *label,* is assigned to each connected component. We denote by $\alpha(K)$ the label of a connected component K of F. The ith *component of* F is the union of the connected components of F that have label i. The ith component may be orientable or not, and it could be empty. We often denote an n-component surface link F by $F_1 \cup \cdots \cup F_n$, where each F_i is the ith component of F. Two n-component surface links F and F' are *unorientedly bordant* if there is a compact 3-manifold $W = \bigcup_{i=1}^n W_i$ properly embedded in $\mathbf{R}^4 \times [0,1]$ such that $\partial W_i = F_i \times \{0\} \cup F_i' \times \{1\}$ for $i = 1, \dots, n$. In this paper, $F \simeq_B F'$ means that F and F' are unorientedly bordant, and $F \cong_A F'$ means that they are ambient isotopic in $\mathbf{R}^4$. The unoriented bordism classes of n-component surface links form an abelian group $UL_{4,n}$ such that the sum $[F] + [F']$ is defined to be the class $[F \amalg F']$ of the split union $F \amalg F'$. The identity is represented by the empty $F = \emptyset$ and the inverse $-[F]$ is represented by the mirror image of F. The following is our main theorem.

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