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On Fixed Points and Determining Sets for Holomorphic Automorphisms

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0. Introduction

It is a result of classical function theory (see [FiF; Les; Mas; PeL; S]) that if $f : U \to U$ is a conformal self-mapping of a plane domain that fixes three distinct points then $f(\zeta) \equiv \zeta$. The purpose of the present paper is to put this result into a geometrically natural context and to extend it to higher-dimensional domains and manifolds. For an examination of fixed point questions from a slightly different point of view, we refer the reader to the work of Vigué (see e.g. [V1; V2]).

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1. Spanning Cartan–Hadamard Subsets

In this section, we let M be a connected, complete Riemannian manifold.

1.1. Cut Points and Cut Loci

Let $x \in M$. A point $y \in M$ is called a *cut point* of x if there are two or more length-minimizing geodesics from x to y in M. We also use the following basic terminology and facts from Riemannian geometry. A geodesic $\gamma : [a, b] \to M$ is called a *length-minimizing geodesic* (or, alternatively, a *minimal geodesic* or a *minimal connector*) from x to y if $\gamma(a) = x$, $\gamma(b) = y$, and $\text{dis}(x, y) =$ arc length of γ. Any two points in a complete Riemannian manifold can be connected by a minimizing geodesic by the Hopf–Rinow theorem. If there is a smooth family of minimizing geodesics from x to y, then these two points are said to be *conjugate.* Conjugate points are cut points. The collection of cut points of x in M is called the *cut locus* of x, which we denote by C_x in this paper. It is known that C_x is nowhere dense in M (see e.g. [GKM; K]).

1.2. Spanning Cartan–Hadamard Sets

A subset X of M is a *Cartan–Hadamard set* if there exists an $x_0 \in X$ such that $X \subset M \setminus C_{x_0}$. We will call x_0 a *pole* of X. A pole of a set is in no way unique. But, for

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