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A Quasi-Paucity Problem

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1. Introduction

A cartographer of the diophantine landscape is compelled to acknowledge the distinguished position occupied by the investigation of diophantine systems in which there are believed to be few other than the obvious solutions. When these systems are symmetric, the task of verifying such a belief has come to be called a *paucity problem*. Although the literature surrounding this topic is by now extensive when the underlying summands are perfect powers (see, for example, the sources recorded in the bibliography), little is known for more general situations. The object of this note is to establish that the number of solutions of certain systems of additive equations is dominated, in essence, by the diagonal contribution alone.

In order to state our main conclusion precisely, we require some notation. Suppose that t is a positive integer, and let $f_1(x), \dots, f_t(x)$ be polynomials with rational coefficients of respective degrees $k_1, \dots, k_t$. When P is a positive number, denote by $S_s(P; \mathbf{f})$ the number of integral solutions of the simultaneous equations

$$\sum_{i=1}^{s} (f_j(x_i) - f_j(y_i)) = 0 \quad (1 \le j \le t), \tag{1}$$

with $1 \le x_i, y_i \le P$ $(1 \le i \le s)$.

Theorem 1. *Suppose that the polynomials* $f_i(x) \in \mathbb{Q}[x]$ $(1 \le i \le t)$ *satisfy the condition that* $1, f_1, \dots, f_t$ *are linearly independent over* $\mathbb{Q}$. *Suppose also that* A *is a positive number sufficiently large in terms of* t, $\mathbf{k}$, *and the coefficients of* $f_1, \dots, f_t$. *Then, whenever* $\max\{k_1, \dots, k_t\} \ge 2$ *and* $P \ge 3$, *one has*

$$S_{t+1}(P; f_1, \dots, f_t) \ll P^{t+1}(\log P)^A.$$

Plainly, those solutions of the system (1) in which $x_1, \dots, x_s$ are simply a permutation of $y_1, \dots, y_s$ provide a contribution to $S_{t+1}(P; \mathbf{f})$ that ensures the lower bound

$$S_{t+1}(P; \mathbf{f}) \ge (t+1)!\, P^{t+1} + O_t(P^t). \tag{2}$$

Thus we may assert that the conclusion of the theorem is somewhat close to a paucity result. We remark that the bound recorded in the theorem was already

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