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Topological Classification of $\mathbf{Z}_p^m$ Actions on Surfaces

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1. Introduction

Let G be a group isomorphic to $\mathbf{Z}_p^m$, where p is a prime integer. Abelian group actions on surfaces constitutes a classical subject (see [E; J1; J2; Na1; N; S; Z]). In [E; J1; J2; Z], a connection is established between the topological equivalence classes of actions and the second homology of the group that is acting. But some attempts to use these results for the classification of abelian actions give wrong results in some cases (cf. [E, Rem. 4.5] with Corollary 12 in our Section 4). The full classification has been found in the cyclic case by Nielsen in [N] and for $\mathbf{Z}_2^m$ in [Na1]. In this paper we present a direct and complete way to deal with the topological classification of $\mathbf{Z}_p^m$ actions, where p is a prime integer ($\mathbf{Z}_p^m = \mathbf{Z}_p \oplus \overset{m}{\cdots} \oplus \mathbf{Z}_p$ and $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$). The main idea of our work is the fact that a fixed point–free action of $\mathbf{Z}_p^m$ provides an alternating bilinear form on $\mathbf{Z}_p^m$.

We give the full description of strong equivalence classes, in particular. In the case of fixed point–free actions, every action of G on a surface define an alternating bilinear form $(\cdot,\cdot)\colon G^* \times G^* \to \mathbf{Z}_p$, where G^* is the group of forms of G on $\mathbf{Z}_p$ (see Definition 7). Two actions of G are strongly equivalent if and only if the actions define the same bilinear form (Theorem 8). All possible such actions are described in Theorem 9. The case of actions having elements with fixed points is considered in Theorems 13 and 14.

Since G is a finite group, it is possible—given an action $(\tilde{S}, f)$ of G—to construct an analytic structure on $\tilde{S}$ such that $f(G)$ is a group of automorphisms of $\tilde{S}$. Hence all the actions considered in this paper appear as automorphism group actions of complex algebraic curves.

A motivation for our study is the description of the set of connected components in the moduli space $M^{p,m}$ of pairs (C, G), where C is a complex algebraic curve and $G \cong \mathbf{Z}_p^m$ is a group of automorphisms of C. According to [Na2], the description of connected components of $M^{p,m}$ is reduced to the description of topological classes of pairs $(\tilde{S}, K)$, where K is a group of autohomeomorphisms of $\tilde{S}$ and where K is isomorphic to $\mathbf{Z}_p^m$. We consider $(\tilde{S}, K)$ and $(\tilde{S}', K')$ to be equivalent if there exists a homeomorphism $\varphi\colon \tilde{S} \to \tilde{S}'$ such that $K' = \varphi \circ K \circ \varphi^{-1}$. These equivalence classes are in one-to-one correspondence with classes of weak

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