TAMING POLYHEDRA IN THE TRIVIAL RANGE

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1. INTRODUCTION

One of the major problems in topology is that of determining the equivalence classes in the set of imbeddings of one topological space into another, under some suitable definition of equivalence of two imbeddings. In the special case in which the domain space is a polyhedron of dimension k and the imbedding space is a combinatorial n-manifold (without boundary), it has been shown [1], [11] that whenever $2k + 2 \leq n$, then any two "sufficiently close" piecewise linear imbeddings are equivalent by an ambient isotopy. Gluck [8], [9], Greathouse [10], and Homma [12] showed that under the same conditions, each locally tame imbedding is equivalent to a piecewise linear imbedding by a homeomorphism of the manifold onto itself. Indeed, Gluck [8], [9] went further to show that this equivalence can actually be realized by an ambient isotopy.

The purpose here is to show that for $2k + 2 \leq n$, the set of locally tame imbeddings of a k-dimensional polyhedron into a combinatorial n-manifold is actually larger than it appears to be at first. The following is our main theorem.

THEOREM 1. *Suppose* f *is an imbedding of a* k*-dimensional polyhedron* X^k *into a combinatorial* n*-manifold* M^n, *where* $2k + 2 \leq n$, *and* P *is a tame polyhedron in* M^n, *with* $\dim P \leq \frac{n}{2} - 1$, *such that* $F \mid (X^k - f^{-1}(P))$ *is locally tame. Then* f *is* ε*-tame.*

2. DEFINITIONS

A k*-dimensional polyhedron* is the underlying space of some finite k-dimensional simplicial complex; it will usually be denoted by X^k. A *combinatorial* n*-manifold* is a locally finite simplicial complex for which the link of each vertex is combinatorially equivalent to the boundary of the standard n-simplex (that is, we shall only consider manifolds without boundary), and it will usually be denoted by M^n.

An imbedding f of X^k into M^n is said to be *tame* if there exists a homeomorphism h of M^n onto itself such that hf: $X^k \to M^n$ is piecewise linear. An imbedding f of X^k into M^n is *locally tame at a point* x *of* X^k if there exists a polyhedral neighborhood Z of x in X^k such that $f \mid Z$ is tame. An imbedding will be called ε-tame if for each $\varepsilon > 0$ there exists an ε-push h of $(M^n, f(X^k))$ (see Section 3) such that hf: $X^k \to M^n$ is piecewise linear.

3. DENSE AND SOLVABLE SETS OF IMBEDDINGS

The techniques employed to prove Theorem 1 are based upon the following notions, introduced by Gluck in [8].

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