ON AUTOMORPHISMS OF A LOCALLY COMPACT GROUP

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In this note, which is a continuation of the paper [5], we show that a continuous automorphism T of a locally compact, noncompact, connected group G is not ergodic. We thus answer in part a question raised by Halmos on page 29 of his book [1]. The result has also been announced by T. S. Wu [6].

We follow [1] for concepts in ergodic theory, and [2] for results on Lie groups and Lie algebras. If G is a connected Lie group, we denote by L its Lie algebra. The elements of the Lie algebra L are denoted by $\dot{x}$, $\dot{y}$, $\cdots$. If T is a continuous automorphism of a connected Lie group G, then the induced automorphism of L is denoted by $\dot{x} \to \dot{x}^t$ or by t.

LEMMA. *Let* G *be a connected Lie group, and let* T *be a continuous automorphism of* G. *Suppose* T *is ergodic. Then* 1 *is the unique characteristic root of* Ad g, *for each element* g *of* G. *Consequently, the associated Lie algebra* L *of* G *is nilpotent.*

Proof. From the identity $(Tg)h(Tg^{-1}) = T(g(T^{-1}h)g^{-1})$ we see that $\mathrm{Ad}\,(Tg) = t(\mathrm{Ad}\,g)t^{-1}$ for all g in G. So, for every complex number λ, the function $\det(\lambda I - \mathrm{Ad}\,g)$ is continuous and T-invariant. Hence it is constant for each λ. So $\lambda = 1$ is its only zero. Therefore Ad $\dot{x}$ is nilpotent for each $\dot{x}$ belonging to L, and L is nilpotent.

THEOREM. *Let* G *be a connected, locally compact group, let* T *be a continuous automorphism of* G, *and let* T *be ergodic. Then* G *is compact.*

Proof. Suppose that G is a Lie group of dimension 1. Then it is abelian, and hence compact, by Theorem 3 of [5]. Now suppose that the theorem is true for all Lie groups of dimension less than n. Let G be a Lie group of dimension n. By the lemma, G is nilpotent. Therefore its centre Z has positive dimension and is invariant under T. Therefore T determines a continuous automorphism T^* of G/Z, in the standard way.

We claim that T^* is ergodic. To prove this, let us take a T^*-invariant Borel subset H^* of G/Z. Let H be the union of cosets in H^*. Then H is a T-invariant Borel subset of G, and by hypothesis, either H or $G \sim H$ has measure 0. The same is true for either H^* or $G/Z \sim H^*$, by the formula for integration on coset spaces [3, p. 131]. Since G/Z is a Lie group of dimension less than n, it is compact, by the induction hypothesis.

Now the group Ad G is completely reducible, since G/Z is compact. Therefore, by the lemma, $\mathrm{Ad}\,G = \{I\}$. Therefore G is abelian, and hence G is compact, by Theorem 3 of [5]. Thus we have proved the theorem in case G is a Lie group.

Now suppose G is merely known to be a locally compact, connected group. Then it contains a maximal compact, normal subgroup N such that G/N is a Lie group [4, p. 172]. Clearly, N is invariant under T. Hence, as in the proof of the case of a Lie group above, T determines an ergodic automorphism T^* of G/N. Therefore G/N is compact, by what was shown above; hence, G is compact.

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