ON ORDER-PRESERVING EXTENSIONS TO REGRESSIVE ISOLS

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1. INTRODUCTION

The extension of functions to isols was treated by Nerode in [4]. If we restrict our attention to regressive isols, the notion of an infinite series of isols, defined in [2], becomes a useful tool. In [1] and [5], such series were employed to study extensions. In particular, Barback proved that for a recursive function f, $f_\Lambda: \Lambda_R \rightarrow \Lambda_R$ if and only if f is eventually increasing. Our main concern here is the determination of the functions that are recursive and eventually increasing and have the additional property that their extensions ultimately preserve the partial ordering $\leq$ in Λ_R. We call the extension f_Λ of a recursive, eventually increasing function f *ultimately order-preserving in* Λ_R if there exists a natural number k such that f_Λ preserves the order $\leq$ in the class of regressive isols that are greater than or equal to k. Our principal result states that among the recursive, eventually increasing functions, those whose extensions are ultimately order-preserving in Λ_R are exactly the functions whose first difference is eventually increasing. Our notation and terminology is that of [5].

2. A THEOREM ON INFINITE SERIES

By a number-theoretic function, we mean any function defined on the nonnegative integers and having integral values. A number-theoretic function is said to be *recursive* if its positive and negative parts are both recursive. We recall the definitions of two additional concepts, defined in [5]: the mapping ϕ_f and the star-sum. If T is an infinite regressive isol and f is a one-to-one function, then $\phi_f(T) = \text{Req}\ \rho t_{f(n)}$, where t_n is any regressive function ranging over any set in T. The star-sum is defined as follows. If f is a recursive, number-theoretic function and T is an infinite, regressive isol, then

$$\sum_T^* f_n = \sum_T f_n^+ - \sum_T f_n^-,$$

where f_n^+, f_n^- are respectively the positive and negative parts of f.

THEOREM 1. *Let* a_n *be a recursive function. Then for all regressive isols* T *and* U

$$\left[U \leq T \Rightarrow \sum_U a_n \leq \sum_T a_n \right] \Leftrightarrow a_n \textit{ is eventually increasing.}$$

Proof. Proceeding from right to left, we first assume that a_n is recursive and eventually increasing. If U is finite, the left-hand side clearly holds. Suppose U is infinite. We first dispense with the case where a_n is increasing. If $U \leq T$, then $U = \phi_f(T)$ for some strictly increasing but not necessarily recursive function f. Thus for this f, $\phi_f(T) \leq T$, and it follows that

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