REMARKS CONCERNING DOMAINS OF SMIRNOV TYPE

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A Jordan domain G with rectifiable boundary is said to be a *Smirnov domain* (or *domain of Smirnov type*) if the derivative of the mapping function f(z) from $|z| < 1$ onto G is an "outer" function, that is, if $\log|f'(re^{i\theta})|$ is the Poisson integral of the boundary values $\log|f'(e^{it})|$. For a fuller discussion of this property, which is of decisive significance for many questions regarding function theory in G, the reader is referred to Privalov [13, p. 159]. A number of sufficient conditions (in terms of geometrical properties of G) that G be of Smirnov type are known, although the picture is far from complete. Some criteria of this kind, due to Smirnov, Keldysh, and Lavrentiev, are given in [13]. More recently, Tumarkin [15] has established a powerful sufficiency criterion. In [7], the concept of Smirnov domain was extended to multiply connected domains; but here we shall be concerned only with Jordan domains.

The main result achieved by Tumarkin in [15] may be roughly expressed this way: if every point of the boundary F of G, with countably many exceptions, is contained in a "nice" arc of F, then G is of Smirnov type (for a more precise formulation, see below). A weaker criterion of this kind was given in [10], the authors being at the time unaware of Tumarkin's results. The method employed in [10] may however be extended to yield a stronger result in which the allowable set of "bad" boundary points need not be countable, provided it is "sparse" in a certain precise metric sense. Section 1 of the present paper is devoted to this result. (Actually, in Theorem 1 we prove a slightly more general result, which does not presuppose rectifiability of the boundary.)

Section 2 of the present paper is essentially independent of Section 1, the only common link being that the remarkable type of pathological curve (which we call a pseudocircle) discussed there was first constructed by Keldysh and Lavrentiev to show that there exist domains not of Smirnov type. Pseudocircles may be characterized independently of conformal mapping (see Theorem 2). The direct construction of some pseudocircle without function theoretic methods would be of interest; but we have not been able to achieve it. Section 2 summarizes the known information concerning pseudocircles.

Notation and conventions. In this paper, D shall always denote the unit disk $|z| < 1$, and C its boundary. An (open) *smooth arc* denotes a homeomorphic image of the interval $0 < t < 1$ by a complex-valued function w(t) having a continuous derivative that is different from zero for all t $(0 < t < 1)$. This is equivalent to saying that the oriented arc parametrized by w(t) admits a continuously varying unit tangent vector. Concerning H^p-spaces and general function-theoretic background, the reader may consult [13]. An *inner function* or *function of class* U *in the sense of Seidel* is a function f of class H^∞ in D with $|f(e^{i\theta})| = 1$ a.e. The *outer factor* of an $f \in H^p$ is the quotient of f by its inner factor in the canonical factorization, and f is an *outer function* if it is equal to its outer factor. (The terms "inner" and "outer," introduced by Beurling, are not particularly suggestive; but they correspond to a distinction that is vital in many questions.)

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