ON A BOUNDARY PROPERTY OF CONTINUOUS FUNCTIONS

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Let D be the open unit disk in the plane, and let C be its boundary, the unit circle. If x is a point of C, then an *arc at* x is a simple arc γ with one endpoint at x such that $\gamma - \{x\} \subset D$. If f is a function defined in D and taking values in a metric space K, then the *set of curvilinear convergence* of f is

$$\{x \in C \mid \text{there exists an arc } \gamma \text{ at } x \text{ and there exists a point } p \in K \text{ such that } \lim_{\substack{z \to x \\ z \in \gamma}} f(z) = p\}.$$

J. E. McMillan proved that if f is a continuous function mapping D into the Riemann sphere, then the set of curvilinear convergence of f is of type $F_{\sigma\delta}$ [2, Theorem 5]. In this paper we shall provide a simpler proof of this theorem than McMillan's, and we shall give a generalization and point out some of its corollaries.

Notation. If S is a subset of a topological space, $\overline{S}$ denotes the closure and S^* denotes the interior of S. Of course, when we speak of the interior of a subset of the unit circle, we mean the interior relative to the circle, not relative to the whole plane. Let K be a metric space with metric ρ. If $x_0 \in K$ and $r > 0$, then

$$S(r, x_0) = \{x \in K \mid \rho(x, x_0) < r\}.$$

An arc of C will be called *nondegenerate* if and only if it contains more than one point.

LEMMA 1. *Let* $\mathscr{I}$ *be a family of nondegenerate closed arcs of* C. *Then* $\bigcup_{I \in \mathscr{I}} I - \bigcup_{I \in \mathscr{I}} I^*$ *is countable.*

Proof. Since $\bigcup_{I \in \mathscr{I}} I^*$ is open, we can write $\bigcup_{I \in \mathscr{I}} I^* = \bigcup_n J_n$, where $\{J_n\}$ is a countable family of disjoint open arcs of C. If

$$x_0 \in \bigcup_{I \in \mathscr{I}} I - \bigcup_{I \in \mathscr{I}} I^*,$$

then for some $I_0 \in \mathscr{I}$, x_0 is an endpoint of I_0. For some n, $I_0^* \subset J_n$, so that $x_0 \in \overline{J}_n$. But $x_0 \notin J_n$, so that x_0 is an endpoint of J_n. Thus $\bigcup_{I \in \mathscr{I}} I - \bigcup_{I \in \mathscr{I}} I^*$ is contained in the set of all endpoints of the various J_n; this proves the lemma. ∎

In what follows we shall repeatedly use Theorem 11.8 on page 119 in [3] without making explicit reference to it. By a cross-cut we shall always mean a cross-cut of D. Suppose γ is a cross-cut that does not pass through the point 0. If V is the component of $D - \gamma$ that does not contain 0, let $L(\gamma) = \overline{V} \cap C$. Then $L(\gamma)$ is a nondegenerate closed arc of C.

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