BOUNDARY PROPERTIES OF FUNCTIONS CONTINUOUS IN A DISC

J. E. McMillan

1. INTRODUCTION

Let f be a continuous function whose domain is the open unit disc D in the complex z-plane and whose range is on the Riemann sphere Ω. A simple continuous curve β: z(t) $(0 \leq t < 1)$ contained in D is called a *boundary path* if $|z(t)| \to 1$ as $t \to 1$. The *end* of a boundary path β is the intersection of the closure $\bar{\beta}$ of β and the circumference C of D. A boundary path β: z(t) $(0 \leq t < 1)$ is an *asymptotic path* of f for the value $a \in \Omega$ provided $f(z(t)) \to a$ as $t \to 1$. The point $a \in \Omega$ is called an *asymptotic value* of f if there exists an asymptotic path of f for the value a, and a is said to be a *point asymptotic value* of f if there exists as asymptotic path of f for the value a whose end consists of a single point of C.

Section 2 is devoted to proving that the set of asymptotic values of f and the set of point asymptotic values of f are analytic sets in Ω (Theorems 2 and 4). Mazurkiewicz [10] proved that the set of asymptotic values of a meromorphic function f in D (or in the plane) is an analytic set, by considering the completion of the "Mazurkiewicz metric" on the Riemann surface of f. We define a distance between sets of "equivalent asymptotic paths" of the continuous function f, and we prove (Theorem 1) that the metric space thus obtained is separable and complete. We then obtain Theorem 2 in the manner of Mazurkiewicz [10]. A more involved application of Theorem 1 is needed for the proof of Theorem 4.

We call the set

$$\{\zeta \in C\text{: there exists an asymptotic path of f with end } \zeta\}$$

the *set of curvilinear convergence* of f. (We sometimes find it convenient to ignore the distinction between $\{\zeta\}$ and ζ.) In Section 3 we prove that it is an $F_{\sigma\delta}$-set (Theorem 5).

Let A be the set of curvilinear convergence of f. A function ϕ whose domain is A and whose range lies on Ω is called a *boundary function* of f if for each $\zeta \in A$ some asymptotic path of f for the value $\phi(\zeta)$ has the end ζ. The investigation of the boundary functions for the case where A = C was initiated by Bagemihl and Piranian [2]. In Section 4 we prove that if ϕ is a boundary function of f, then there exists a function of Baire class 1 on A that differs from ϕ at only countably many points of A (ϕ is of honorary Baire class two), and thus in particular that ϕ is of Baire class two on A. Hence we generalize a recent theorem of Kaczynski [6, p. 596] who considered the case where A = C.

2. THE SETS OF ASYMPTOTIC VALUES

Let $\chi(a, b)$ denote the three-dimensional Euclidean distance between the points a and b of Ω. Then $\chi(a, b) \leq 1$ $(a, b \in \Omega)$. By a *rational disc* we mean a set of the form

Received November 5, 1965.