AN ELEMENTARY PROOF OF THE POWER INEQUALITY FOR THE NUMERICAL RADIUS

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Let $\mathscr{H}$ be a complex Hilbert space, finite- or infinite-dimensional. With each bounded linear operator A on $\mathscr{H}$ there is associated the nonnegative number

$$w(A) = \sup_{\|x\|=1} |(Ax, x)|,$$

called the *numerical radius* of A. Many properties of the function w are well known and quite elementary. Among these are the following:

(1) $w(A) = 0$ if and only if $A = 0$;

(2) $w(\lambda A) = |\lambda| \, w(A)$ for every scalar λ;

(3) $w(A + B) \leq w(A) + w(B)$;

(4) $\frac{1}{2} \|A\| \leq w(A) \leq \|A\|$;

(5) $r(A) \leq w(A)$, where $r(A)$ denotes the spectral radius of A;

(6) w is not continuous in either the weak or the strong operator topology, but is continuous in the uniform operator topology.

When we inquire into the multiplicative properties of w, the inequality

$$(*) \qquad w(AB) \leq w(A)\,w(B)$$

naturally comes to mind. It is easy to see that (*) cannot hold universally, and perhaps somewhat more surprising that (*) can fail for commutative A and B. In fact, the following example, pointed out by Arlen Brown and Allen Shields, shows that (*) can fail when A and B are powers of the same operator. Let N denote the nilpotent operator on a 4-dimensional space whose matrix relative to some orthonormal basis is

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}.$$

Easy calculations show that $w(N) < 1$, $w(N^2) = 1/2$, and $w(N^3) = 1/2$. Thus (*) fails with $A = N$ and $B = N^2$.

In spite of all this unpleasantness, the following theorem is true.

THEOREM. *If* A *is an operator on* $\mathscr{H}$, *and* n *is a positive integer, then*

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