WEYL'S THEOREM FOR NONNORMAL OPERATORS

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1. INTRODUCTION

Let $\mathscr{B}(H)$ be the algebra of all bounded operators on an infinite-dimensional complex Hilbert space H, and let $\mathscr{K}$ be the closed ideal of compact operators. I write $\sigma(A)$ for the spectrum of A in $\mathscr{B}(H)$, and I define the Weyl spectrum $\omega(A)$ by

$$\omega(A) = \bigcap \sigma(A+K),$$

where the intersection is taken over all K in $\mathscr{K}$. A celebrated theorem of Weyl [7] asserts that if A is normal ($A^*A = AA^*$), then $\omega(A)$ consists precisely of all points in $\sigma(A)$ except the isolated eigenvalues of finite multiplicity.

In this paper, I show that Weyl's theorem holds for two large classes of generally nonnormal operators. The first of these is the class of hyponormal operators, which has been studied in [6]. The second class of operators for which Weyl's theorem holds is the class of Toeplitz operators, which has been studied in [2], [3], [8] and in many other papers. (References to much of the pertinent literature can be found in [2].) Finally, the present study of hyponormal operators and Toeplitz operators suggests a notion of "extremally noncompact" operators, which I examine in the last part of this paper.

2. PRELIMINARIES

Recall that an operator A is a Fredholm operator if its range R(A) is closed and both $R(A)^\perp$ and the null space N(A) are finite-dimensional. The Fredholm operators $\mathscr{F}$ constitute a multiplicative open semigroup in $\mathscr{B}(H)$. In fact [1], if π is the natural quotient map from $\mathscr{B}(H)$ to $\mathscr{B}(H)/\mathscr{K}$, then A is in $\mathscr{F}$ if and only if $\pi(A)$ is invertible. For any A in $\mathscr{F}$, the index i(A) is defined by the formula

$$i(A) = \dim N(A) - \dim R(A)^\perp,$$

and it is known that i is a continuous integer-valued function on $\mathscr{F}$.

Schechter [5] has observed that for any operator A,

$$\omega(A) = \{\lambda \mid A - \lambda \notin \mathscr{F}\} \cup \{\lambda \mid A - \lambda \in \mathscr{F} \text{ and } i(A-\lambda) \neq 0\},$$

and I shall use this characterization of $\omega(A)$. Note that by Schechter's result, $\omega(A)$ is never empty, since

$$\{\lambda \mid A - \lambda \notin \mathscr{F}\} = \sigma[\pi(A)].$$

It should also be noted that

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