FUNCTIONS OF BOUNDED CHARACTERISTIC WITH PRESCRIBED AMBIGUOUS POINTS

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Let $f(z)$ be a complex-valued uniform function defined for $|z| < 1$. We shall call a point ζ an *ambiguous point* for $f(z)$, if $|\zeta| = 1$ and there exist two Jordan arcs J_1 and J_2, terminating in ζ and lying, except for ζ, in $|z| < 1$, such that

$$\lim_{\substack{z \to \zeta \\ z \in J_1}} f(z) \quad \text{and} \quad \lim_{\substack{z \to \zeta \\ z \in J_2}} f(z)$$

both exist and are unequal.

It has recently been shown [1, p. 382] that even if no further conditions are imposed on $f(z)$, there are at most enumerably many ambiguous points for $f(z)$, and it is well-known [2, p. 66] that if $f(z)$ is regular and bounded in $|z| < 1$, there are no ambiguous points for $f(z)$. On the other hand, corresponding to every enumerable set E on $|z| = 1$ there exist [1, p. 381] regular functions in $|z| < 1$ for which every point of E is an ambiguous point, and it is thus natural to ask whether such regular functions can, in some sense, be "nearly" bounded. Now, regular functions of bounded characteristic possess [2, pp. 185, 208, 209] some of the important boundary properties of bounded regular functions. The following result shows, however, that the two classes of functions are quite different in respect to the existence of ambiguous points.

THEOREM. *Let*

$$E = \{\zeta_1, \zeta_2, \cdots, \zeta_n, \cdots\}$$

be an enumerable set of points on $|z| = 1$. *Then there exists a function* $f(z)$, *regular and of bounded characteristic in* $|z| < 1$, *for which every element of* E *is an ambiguous point.*

Proof. A function $g(z)/h(z)$, where $g(z)$ and $h(z)$ are bounded and regular in $|z| < 1$ and $h(z) \neq 0$ in $|z| < 1$, is a regular function of bounded characteristic in $|z| < 1$ [2, p. 189]; we shall obtain an $f(z)$ which is of this form and satisfies the conclusion of the theorem. To this end, it is obviously sufficient to construct $g(z)$ and $h(z)$ so that they satisfy the following conditions:

(I) *There exists a constant* $b > 0$ *such that, in* $|z| < 1$,

$$|g(z)| > b(1 - |z|)^2,$$

and $g(z) \to 0$ *as* z *tends to an arbitrary point of* E.

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