SOME POINT SETS ASSOCIATED WITH TAYLOR SERIES

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1. INTRODUCTION

Let P be a property which a function f(z) (regular in $|z| < 1$) or its Taylor series $\Sigma a_n z^n$ may or may not possess at a point $z = e^{i\theta}$ on the unit circle C. And let K(P) be the class of point sets E on C for which there exists a function $f = f_E$ (again regular in $|z| < 1$) possessing the property P at each point of E and at no point of C — E. In the case where P is the property of convergence of the Taylor series of f, Sierpiński [11] (see also Steinhaus [13, pp. 435-436]) pointed out that for reasons of cardinality the class K(P) cannot contain every set on C: indeed, there are 2^c point sets on C, but only $c^{\aleph_0} = c$ analytic functions f; in the application of this argument it is, of course, quite irrelevant what the property P may be.

The problem of obtaining a characterization of the sets of convergence of Taylor series was attacked with considerable success by Lusin [7], Steinhaus [13] and [14], Neder [10], and Mazurkiewicz [9]. Recently, further advances have been made by the present authors, partly in collaboration with Paul Erdös [1], [4], [5]. In particular, it was shown in [4, I] that K(P) contains every set of type F_σ, that is, every set which can be represented as the union of countably many closed sets. The same is true (see [5]) if P denotes the property that the function f possesses a radial limit on the radius vector of the point $e^{i\theta}$.

In the present paper, we consider the problem of determining the class K(P) in the case in which P denotes boundedness of the sequence $\{s_m(e^{i\theta})\}$, where $s_m(z) = \Sigma_{k=0}^{m} a_k z^k$; also, in the case in which P denotes boundedness of f on the radius $z = re^{i\theta}$ $(0 \leq r < 1)$. In both cases, the problem is amenable to complete solution: K(P) consists of all sets of type F_σ.

In both of these instances, the proof that every set of type F_σ on C is in K(P) could be obtained by appropriate modifications of the construction in [5]. However, we prefer giving a proof based on a new construction. This construction avoids the appeal which had previously been made to Hermite's and Hurwitz's theorem on the approximation of real numbers by rationals (see [3] and [6]), and it is simpler in detail than the earlier constructions. Its simplicity is due largely to the introduction of the polynomials $[(z^n - 1)/(z - 1)]^2$; the substitution of these functions for Lusin's polynomials $(z^n - 1)/(z - 1)$, which had previously been used, was suggested by Paul Erdös in a conversation on sets of convergence.

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