ON A CONJECTURE OF LUSIN

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1. INTRODUCTION

A point $e^{i\theta}$ will be called a Lusin point of the function f(z) provided f is holomorphic in $|z| < 1$ and maps every disc $|z - te^{i\theta}| < 1 - t$ $(0 < t < 1)$ upon a region (possibly many-sheeted) of infinite area. With this terminology, a conjecture of Lusin [2] may be stated as follows: *There exists a bounded function for which every point of* $|z| = 1$ *is a Lusin point.* Recently, Kufarev and Semukhina [1] have shown that the set of Lusin points of a bounded function can be everywhere dense on $|z| = 1$. In Section 2, we shall prove that there exists a function which is continuous in $|z| \leq 1$ and for which every point $e^{i\theta}$ of $|z| = 1$ is a Lusin point. Our method consists in proving that the function

$$(1) \qquad \Sigma a_k z^{n_k} \quad (a_k \neq 0;\ k = 1, 2, \cdots)$$

has every point $e^{i\theta}$ as a Lusin point, provided $n_k \to \infty$ rapidly enough; in this statement, the expression "rapidly enough" must of course be interpreted in terms of the sequence $\{a_k\}$. The result is somewhat related to theorems of Salem and Zygmund [3] and of Schaeffer [4], who showed that if the series $\Sigma |a_k|$ converges slowly enough and $n_k \to \infty$ rapidly enough, the function (1) maps the circle $|z| = 1$ into a Peano curve. Intuitively, this proposition is suggested by the fact that, for $|a| < 1$ and large n, the polynomial $z + az^n$ maps the unit circle into a curve which consists of $n - 1$ nearly circular loops, of radius $|a|$, whose "moving center" lies on the unit circle.

In Section 3 we show that a function f, holomorphic in $|z| < 1$, continuous in $|z| \leq 1$, and mapping the unit disc onto a region of infinite area, need not possess any Lusin points at all. Our proof is based on the construction of a function which maps the unit disc upon a roughly circular disc to which many small discs are attached. The function has the further property that it takes no value infinitely often, for $|z| \leq 1$. (In a conversation, Professor K. Noshiro had raised the question whether there exists a function $\Sigma a_n z^n$, continuous in $|z| \leq 1$, with $\Sigma n |a_n|^2 = \infty$, and taking no value infinitely often in $|z| \leq 1$. The referee has pointed out a very simple alternate construction of such a function: let the Riemann surface R consist of a ribbon which covers

once	the annulus	$0 < \lvert w\rvert < 1/2$,
4 times	the annulus	$0 < \lvert w - 1/2\rvert < 1/4$,
16 times	the annulus	$0 < \lvert w - 3/4\rvert < 1/8$,

and so forth; and let f map the unit disc upon R.)

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