A REMARK ON DIFFERENTIABLE MAPPINGS

Rolf Nevanlinna

1. We consider a function $y = y(x)$ which maps the sphere $|x| < R$ $(R \leq \infty)$ of the n-dimensional euclidean space L_n $(n \geq 2)$ onto a domain G of the same space. Let G_r be the image of the sphere $|x| \leq r$ $(r < R)$. If $V(r)$ denotes the volume of the domain G_r and $A(r)$ the measure of its boundary Γ_r, then we have, by the isoperimetric inequality,

$$n^n v_1 V^{n-1} \leq A^n, \tag{1}$$

where v_1 denotes the volume of the unit sphere (equal to $\pi^k/k!$ for $n = 2k$, and to $2(2\pi)^{k-1}/1\cdot 3 \cdots (2k-1)$ for $n = 2k - 1$).

2. We suppose that the derivative operator $y'(x) \equiv \frac{dy(x)}{dx}$ is continuous. The jacobian $\Delta(x)$ of the function $y(x)$ is then also continuous. Let $d\sigma_x$ be the measure of an $(n - 1)$-dimensional element of the sphere $|x| = r$, and $d\sigma_y$ the measure of its image. By the Hölder inequality, we have

$$A^n = \left(\int_{|x|=r} \left(\frac{d\sigma_y}{d\sigma_x} \Delta^{\frac{1}{n} - 1} \right) \Delta^{1 - \frac{1}{n}} d\sigma_x \right)^n$$

$$\leq \int_{|x|=r} \left(\frac{d\sigma_y}{d\sigma_x} \right)^n \Delta^{1-n} d\sigma_x \cdot \left(\int_{|x|=r} \Delta \, d\sigma_x \right)^{n-1}.$$

It follows now from (1) that

$$n^n v_1 V^{n-1} \leq \int_{|x|=r} \left(\frac{d\sigma_y}{d\sigma_x} \right)^n \Delta^{1-n} d\sigma_x \left(\int_{|x|=r} \Delta \, d\sigma_x \right)^{n-1}. \tag{2}$$

3. Here obviously

$$\int_{|x|=r} \Delta \, d\sigma_x = \frac{dV}{dr}. \tag{3}$$

The first right-hand integral in (2) has the following geometrical meaning. Let $|x - x_0| \leq r$ be an infinitesimal n-dimensional sphere and E_n the corresponding ellipsoid. If E_n has the semi-axes $a_1 \leq a_2 \leq \cdots \leq a_n$, then the jacobian takes the form

$$\Delta = \frac{a_1 \cdots a_n}{r^n}.$$

Received April 29, 1955.