ON POWER SERIES DIVERGING EVERYWHERE ON THE CIRCLE OF CONVERGENCE

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1. Lusin [4] (see also Dienes [1, pp. 463, 464] or Landau [3, §15]) constructed a power series

(1) $$\sum_{n=0}^{\infty} a_n z^n$$

which satisfies the condition

(2) $$\lim_{n \to \infty} a_n = 0$$

and diverges at every point of the unit circle C. Recently, Herzog [2] gave an example of such a series whose coefficients are real, nonnegative, and satisfy not only (2), but even the stronger condition $a_n = O(n^{-1/3})$. The theorem which we are about to state and prove implies the existence of a series (1) which diverges everywhere on C and satisfies, e.g., the condition $0 < a_n < (n \log n)^{-1/2}$ $(n = 3, 4, \cdots)$.

THEOREM 1. *Let* $\{b_n\}$ *be a sequence of complex numbers satisfying the conditions*

(3) $$|b_n| \geq |b_{n+1}| \quad (n = 0, 1, \cdots)$$

and

(4) $$\sum_{n=0}^{\infty} |b_n|^2 = \infty.$$

Then there exists a power series (1), *with*

(5) $$a_n \text{ equal to either } b_n \text{ or } 0 \quad (n = 0, 1, \cdots),$$

which diverges everywhere on C.

The monotonicity condition (3) cannot be entirely dispensed with, since every power series $\sum_1^{\infty} c_n z^{t_n}$ with $c_n \to 0$ and $\sum_1^{\infty} t_n/t_{n+1} < \infty$ converges on a set which is everywhere dense on C. Condition (4) probably cannot be relaxed at all; indeed, it has been conjectured that every power series $\sum b_n z^n$ satisfying (4) converges almost everywhere on C.

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