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On Spherically Convex Univalent Functions

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1. Introduction

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, and let $\mathbb{T} = \partial\mathbb{D}$. A domain G on the Riemann sphere $\hat{\mathbb{C}}$ is called *spherically convex* if, for any pair $w_1, w_2 \in G$, the smaller arc of the greatest circle (spherical geodesic) between w_1 and w_2 also lies in G.

An analytic univalent function g in $\mathbb{D}$ is called *convex* if $g(\mathbb{D})$ is a convex domain in $\mathbb{C}$. A meromorphic univalent function f in $\mathbb{D}$ is called *spherically convex* (s-convex) if $f(\mathbb{D})$ is a spherically convex domain in $\hat{\mathbb{C}}$.

Let $\mathrm{Rot}(\hat{\mathbb{C}})$ denote the group of rotations of the Riemann sphere $\hat{\mathbb{C}}$ that consists of the Möbius transformations

$$\varphi(z) = e^{i\vartheta}(z-a)/(1+\bar{a}z), \quad a \in \mathbb{C}, \ \vartheta \in \mathbb{R}, \tag{1.1}$$

together with $\varphi(z) = e^{i\vartheta}/z$. Let $\mathrm{Möb}(\mathbb{D})$ denote the group of Möbius transformations of $\mathbb{D}$ onto itself. If f is s-convex, then

$$f^* = \varphi \circ f \circ \psi, \quad \varphi \in \mathrm{Rot}(\hat{\mathbb{C}}), \ \psi \in \mathrm{Möb}(\mathbb{D}) \tag{1.2}$$

is again s-convex and we have $f^*(\mathbb{D}) = \varphi(f(\mathbb{D}))$.

The spherical and Schwarzian derivatives

$$f^{\#} = \frac{|f'|}{1+|f|^2}, \qquad S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \tag{1.3}$$

are unchanged if we replace f by $\varphi \circ f$, with $\varphi \in \mathrm{Rot}(\hat{\mathbb{C}})$. We introduce

$$\sigma(f) = \max_{z \in \mathbb{D}} (1-|z|^2) f^{\#}(z). \tag{1.4}$$

It is clear that $\sigma(\varphi \circ f \circ \psi) = \sigma(f)$ for $\varphi \in \mathrm{Rot}(\hat{\mathbb{C}})$ and $\psi \in \mathrm{Möb}(\mathbb{D})$. The quantity $\sigma(f)$ measures the thickness of $f(\mathbb{D})$ and corresponds to the Bloch norm in the Euclidean case (see e.g. [ACP] and [BMY]).

Replacing f by $\varphi \circ f$ with $a = f(0)$ and suitable ϑ in (1.1), we may often assume that our s-convex function f is *normalized*:

$$f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \cdots, \quad 0 < \alpha \le 1; \tag{1.5}$$

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