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L^q-Differentials for Weighted Sobolev Spaces

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1. Introduction

Let $B(x_0, r)$ denote the open ball in $\mathbf{R}^n$ with center x_0 and radius r. Throughout the paper we assume that all measures are Borel and satisfy $0 < \mu(B) < \infty$ for all balls B.

Definition 1.1. Let μ be a measure on $\mathbf{R}^n$. We say that a function u is *differentiable at x_0 in the $L^q(\mu)$ sense* if

$$\lim_{r\to 0} \frac{1}{r}\left(\fint_{B(x_0,r)} |u(x) - u(x_0) - \nabla u(x_0)\cdot(x - x_0)|^q \, d\mu(x)\right)^{1/q} = 0. \tag{1}$$

Here and in what follows, the symbol $\fint$ stands for the mean-value integral

$$\fint_B f\,d\mu = \frac{1}{\mu(B)}\int_B f\,d\mu.$$

For μ equal to the Lebesgue measure, the following theorem about L^q-differentials of Sobolev functions is well known (see e.g. Theorem 12 in Calderón and Zygmund [3] or Theorem 1, Chapter VIII in Stein [14]).

Theorem 1.2. *Let u be a function from the Sobolev space $H^{1,p}(\Omega)$, where $\Omega \subset \mathbf{R}^n$ $(n \ge 2)$ and $1 \le p < n$. Then u is differentiable in the L^q sense with $q = np/(n-p)$ a.e. in Ω. If $p = n$, then the same is true for all $q < \infty$. Moreover, if $u \in H^{1,p}(\Omega)$ and $p > n$, then u can be modified on a set of measure zero so that it becomes differentiable a.e. in Ω in the classical sense.*

Theorem 1.2 can be regarded as a higher-order analog of the classical Lebesgue differentiation theorem: If $u \in L^p_{\mathrm{loc}}(\mathbf{R}^n, \mu)$, $1 \le p < \infty$, and μ is a Radon measure, then μ-a.e. $x_0 \in \mathbf{R}^n$ is an $L^p(\mu)$-Lebesgue point of u; that is,

$$\lim_{r\to 0}\left(\fint_{B(x_0,r)} |u(x) - u(x_0)|^p \, d\mu(x)\right)^{1/p} = 0. \tag{2}$$

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