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Fixed-Point Indices, Homoclinic Contacts, and Dynamics of Injective Planar Maps

Morris W. Hirsch

1. Introduction

Let $f: \mathbf{R}^2 \to \mathbf{R}^2$ be a C^1 diffeomorphism. Let $p \in \mathrm{Fix}(f)$ be a fixed point that is a *direct saddle*; that is, the derivative of f at p has eigenvalues λ, μ such that $0 < \lambda < 1 < \mu$. The Hartman–Grobman theorem implies that there is a topological coordinate system centered at p representing f as a linear map with these eigenvalues (see [13; 20]).

Our main results assert that, when the closures of the stable and unstable curves at p intersect in certain ways, there exists a fixed-point free Jordan curve $J \subset S$ whose index under f (defined below) is 1. Such a curve surrounds a *block* B of fixed points (i.e., B is open and closed in $\mathrm{Fix}(f)$) that is disjoint from p and has fixed point index 1. In particular, *there exists a fixed point different from p*. When fixed points are isolated with negative indices, the dynamics is shown to be rather simple.

For nonplanar surfaces, even a homoclinic point does not guarantee a second fixed point, as shown by the diffeomorphism of the torus $\mathbf{R}^2/\mathbf{Z}^2$ induced by the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Section 3 contains background material and proofs of the main results. Section 4 shows that many results on planar homeomorphisms generalize to homologically nilpotent injective maps in planar surfaces. In particular, the theorems of Brouwer and Brown, as well as the new results, hold for such maps.

Terminology. All maps are assumed to be continuous; diffeomorphisms are C^1 (continuously differentiable). A set X is *forward invariant* for a map g if $gX \subset X$, *overflowing* if $gX \supset X$, and *invariant* if $gX = X$. Homeomorphism is indicated by $\approx$. The set of fixed points of g is denoted by $\mathrm{Fix}(g)$. The *omega limit set* $\omega(x)$ is the limit set of the sequence $\{g^n x\}_{n \in \mathbf{N}}$. The set of natural numbers is $\mathbf{N} = \{0, 1, \dots\}$ and the set of integers is $\mathbf{Z}$. The Euclidean norm of $x \in \mathbf{R}^2$ is denoted $\|x\|$.

2. Background

Brouwer's 1912 plane translation theorem [1] can be stated as follows.

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