Michigan Math. J. 47 (2000)

Geometric Properties of Pluricomplex Green Functions with One or Several Poles in $\mathbb{C}^n$

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0. Introduction and Statement of Results

In this paper we study the infinitesimal behavior near poles and the boundary behavior of pluricomplex Green functions with one or several logarithmic poles.

On the one hand, we prove a min-max principle for the Azukawa pseudometric that is related to the pluricomplex Green function. On the other hand, we find a new proof of effective formulas for the pluricomplex Green function with two poles of equal weights in the unit ball in $\mathbb{C}^2$. With these formulas, we show that the sublevel sets of this function are not (lineally) convex, no matter how close to the boundary they are situated. This fact is surprising, especially since this convexity property is lost in the case of several poles even when the domain is the unit ball (the sublevel sets of the pluricomplex Green function with one pole of a bounded convex domain are always convex). Moreover, this provides a counterexample to a recently published statement.

Let us recall first the definition of the *pluricomplex Green function* with one or several logarithmic poles in a domain D in $\mathbb{C}^n$. Let m be a positive integer and let $P = \{(p_1, c_1), \dots, (p_m, c_m)\}$ be a set of m distinct poles p_j in D with positive weights c_j, $j = 1, \dots, m$. Following Lelong (see [Le1] and [Le2]), the pluricomplex Green function with poles in P is defined on D by

$$g_D(P, z) = \sup\{u(z) : u \in \mathrm{PSH}(D, [-\infty, 0[) \text{ and } u(z) - c_j \log\|z - p_j\| \text{ is bounded from above for } z \text{ near } p_j,\ j = 1, \dots, m\},$$

where $\mathrm{PSH}(D)$ denotes the set of plurisubharmonic (psh) functions on D. If $m = 1$ and $c_1 = 1$, then $g_D(p, \cdot)$ is the well-known pluricomplex Green function with one logarithmic pole in p, introduced by Klimek (see [Kl]). The pluricomplex Green function has a connection to the complex Monge–Ampère operator. This operator acts on locally bounded psh functions (see [BT1] and [BT2]) and it applies also for psh functions u such that $u^{-1}(-\infty)$ is relatively compact (see [C; D1; Ki; Si]).

Let us consider the following Dirichlet problem for the complex Monge–Ampère operator:

Received April 21, 1998. Revision received August 19, 1999.