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A New Characterization of Hyperellipticity

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1. Introduction

A (geodesic) *necklace* on a closed Riemann surface of genus $p \geq 2$ is a cyclically ordered set of $2p + 2$ simple nondividing closed geodesics (in the hyperbolic metric) $L_1, \dots, L_{2p+2}$, where each L_i intersects L_{i-1} exactly once, intersects L_{i+1} exactly once, and is otherwise disjoint from every other geodesic in the necklace. In this note we give a new characterization of hyperellipticity in terms of geodesic necklaces; this characterization is distinct from that given by Schmutz-Schaller [11]. We also give a geometric proof of Jørgensen's theorem [5], which states that, on a hyperbolic orbifold of dimension 2, there are infinitely many closed geodesics passing through every point of intersection of closed geodesics.

We denote the hyperbolic plane by $\mathbb{H}^2$; we will usually regard this as the upper half-plane. The group of all orientation preserving isometries of $\mathbb{H}^2$ can be canonically identified with $\mathrm{PSL}(2, \mathbb{R})$, the group of real 2×2 matrices with unit determinant.

A discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ is *elementary* if it is a finite extension of a cyclic group. For our purposes, a Fuchsian group is a finitely generated nonelementary discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

We will use the following notation throughout. Matrices in $\mathrm{PSL}(2, \mathbb{R})$ are denoted by $\tilde{a}, \tilde{b}, \dots$; the corresponding hyperbolic isometries are denoted by $a, b, \dots$. If the transformation a is hyperbolic, its axis is denoted by A_a; further, if a is a hyperbolic element of the discrete group G, then we denote by L_a the projection of A_a, which is a geodesic on $\mathbb{H}^2/G$.

Elliptic elements of order 2 are called *half-turns*. The fixed point of a half-turn in $\mathbb{H}^2$ is its *center* (or *vertex*). In general, for any group H and for any set A, the *stabilizer* of A in H is given by

$$\mathrm{Stab}(A) = \{h \in H \mid h(A) = A\}.$$

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