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The Topology of Smooth Divisors and the Arithmetic of Abelian Varieties

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In honor of William Fulton

We have three main results. First, we show that a smooth complex projective variety that contains three disjoint codimension-1 subvarieties in the same homology class must be the union of a whole 1-parameter family of disjoint codimension-1 subsets. More precisely, the variety maps onto a smooth curve with the three given divisors as fibers, possibly multiple fibers (Theorem 2.1). The beauty of this statement is that it fails completely if we have only two disjoint divisors in a homology class, as we will explain. The result seems to be new already for curves on a surface. The key to the proof is the Albanese map.

We need Theorem 2.1 for our investigation of a question proposed by Fulton as part of the study of degeneracy loci. Suppose we have a line bundle on a smooth projective variety that has a holomorphic section whose divisor of zeros is smooth. Can we compute the Betti numbers of this divisor in terms of the given variety and the first Chern class of the line bundle? Equivalently, can we compute the Betti numbers of any smooth divisor in a smooth projective variety X in terms of its cohomology class in $H^2(X, \mathbf{Z})$?

The point is that the Betti numbers (and Hodge numbers) of a smooth divisor are determined by its cohomology class if the divisor is ample or if the first Betti number of X is zero (see Section 4). We want to know if the Betti numbers of a smooth divisor are determined by its cohomology class without these restrictions. The answer is "no". In fact, there is a variety that contains two homologous smooth divisors, one of which is connected while the other is not connected. Fortunately, we can show that this is a rare phenomenon: if a variety contains a connected smooth divisor that is homologous to a nonconnected smooth divisor, then it has a surjective morphism to a curve with some multiple fibers, and the two divisors are both unions of fibers. This is our second main result, Theorem 5.1.

We also give an example of two connected smooth divisors that are homologous but have different Betti numbers. Conjecture 6.1, suggested by this example, asserts that two connected smooth divisors in a smooth complex projective variety X that are homologous should have cyclic etale coverings that are deformation equivalent to each other. The third main result of this paper, Theorem 6.3, is that this conjecture holds, in a slightly weaker form (allowing deformations into positive characteristic), under the strange assumption that the Picard variety of X is isogenous to a product of elliptic curves. The statement in general would follow from a well-known open problem in the arithmetic theory of abelian varieties,

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