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Some Real and Unreal Enumerative Geometry for Flag Manifolds

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To Bill Fulton on the occasion of his 60th birthday

Introduction

For us, enumerative geometry is concerned with counting the geometric figures of some kind that have specified position with respect to some fixed, but general, figures. For instance, how many lines in space are incident on four general (fixed) lines? (Answer: 2.) Of the figures having specified positions with respect to fixed *real* figures, some will be real while the rest occur in complex conjugate pairs, and the distribution between these two types depends subtly upon the configuration of the fixed figures. Fulton [12] asked how many solutions to such a problem of enumerative geometry can be real and later with Pragacz [14] reiterated this question in the context of flag manifolds.

It is interesting that, in every known case, *all* solutions may be real. These include the classical problem of 3264 plane conics tangent to 5 plane conics [30], the 40 positions of the Stewart platform of robotics [5], the 12 lines mutually tangent to 4 spheres [24], the 12 rational plane cubics meeting 8 points in the plane [19], all problems of enumerating linear subspaces of a vector space satisfying special Schubert conditions [34], and certain problems of enumerating rational curves in Grassmannians [36]. These last two examples give infinitely many families of nontrivial enumerative problems for which all solutions may be real. They were motivated by recent, spectacular computations [9; 40] and a very interesting conjecture of Shapiro and Shapiro [35], and were proved using an idea from a homotopy continuation algorithm [16; 17].

We first formalize the method of constructing real solutions introduced in [34; 36], which will help extend these reality results to other enumerative problems. This method gives lower bounds on the maximum number of real solutions to some enumerative problems, in the spirit of [18; 38]. We then apply this theory to two families of enumerative problems, one on classical (SL_n) flag manifolds and the other on Grassmannians of maximal isotropic subspaces in an orthogonal vector space, showing that all solutions may be real. These techniques allow us to prove the opposite result—that we may have no real solutions—for a family of enumerative problems on the Lagrangian Grassmannian. Finally, we suggest a further problem to study concerning this method.

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