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Globally F-Regular Varieties: Applications to Vanishing Theorems for Quotients of Fano Varieties

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Dedicated to Professor William Fulton on the occasion of his sixtieth birthday

1. Introduction

A smooth projective variety is said to be *Fano* if its anti-canonical bundle is ample. The Kodaira vanishing theorem easily implies vanishing of all higher cohomology modules of numerically effective line bundles on any Fano variety, at least in characteristic 0. Indeed, for any positive i, it implies that $H^i(X, \mathcal{L}) = H^i(X, (\mathcal{L} \otimes \omega^{-1}) \otimes \omega)$ vanishes when $\mathcal{L} \otimes \omega^{-1}$ is ample, and hence vanishing holds in particular whenever $\mathcal{L}$ is numerically effective and ω^{-1} is ample.

In this paper, a class of algebraic varieties is introduced, the class of *globally F-regular varieties.* Globally F-regular varieties have strong vanishing properties, including the vanishing of the higher cohomology groups for any numerically effective line bundle (as discussed above for Fano varieties). Indeed, the class of globally F-regular varieties of characteristic 0 is shown to include Fano varieties, so the vanishing just described is recovered. A nice feature of the class of globally F-regular varieties is that it is preserved under the operation of forming certain (and conjecturally: any) GIT quotients by linearly reductive groups.

Globally F-regular varieties are closely related to Frobenius split varieties [MRn]. Both Frobenius splitting and global F-regularity are notions defined using the Frobenius morphism in characteristic p; by reduction to characteristic p, both Frobenius splitting and global F-regularity make sense in characteristic 0 as well. As explained within, global F-regularity turns out to be a stable version of the notion of *Frobenius split along a divisor* that has arisen in the Indian school of algebraic groups [MRn; RR; R1; R2]. However, the definition of global F-regularity is based on the theory of tight closure introduced by Hochster and Huneke in [HH1]: roughly speaking, a projective algebraic variety is globally F-regular if it has a coordinate ring in which all ideals are tightly closed.

The original motivation for this work was a question of Allen Knutson in his study [Kn] of torus actions in symplectic geometry: Let G be a semi-simple complex algebraic group with fixed Borel subgroup B and maximal torus $T \subset B$. Consider the geometric invariant theory (GIT) quotient X of the homogeneous space G/B with respect to some choice of linearization of the natural left action

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