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The Hirzebruch–Riemann–Roch Theorem

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Dedicated to Professor William Fulton on his sixtieth birthday

It is indeed an honor to dedicate this essentially self-contained proof of HRR to William Fulton, whose contributions to the study of Chow groups, intersection theory, and the Riemann–Roch theorems have led to a deeper understanding of these topics.

As is well known, Grothendieck formulated a relative version GRR of the Riemann–Roch for proper morphisms $f: X \to Y$, and HRR turned out to be the special case when Y is a point. To prove GRR, Grothendieck showed that it sufficed to prove GRR for the projection $Y \times \mathbb{P}^n \to Y$ and for a closed immersion. The former is easy, but the latter is much more subtle; in parallel, Grothendieck also proved the Chern character induces an isomorphism

$$\mathrm{ch}: \mathbb{Q} \otimes K(X) \to \mathbb{Q} \otimes A(X).$$

Accounts of Grothendieck's method are found in [SGA; BS; M]. Fulton proved GRR *without denominators* for closed immersions quite directly by the famous "degeneration to the normal cone". This method has since been used in several related contexts (see e.g. [Fa] and [GS]).

The aim of this note is to give a direct proof of HRR that does not rely of Grothendieck's method of factoring a morphism. What is crucially used here, however, is the formalism introduced by Grothendieck, and in particular the isomorphism $K(X) \to \mathcal{G}(X)$ of the K-groups of vector bundles and coherent sheaves, respectively, when X is regular (the hypothesis of quasi-projectivity was removed by Kleiman; see [F]). Our method can be extended to deduce GRR itself directly, but this has not been carried out here.

The HRR for compact complex manifolds was deduced by Atiyah and Singer from their index theorem; it was also proved by methods of differential geometry by Patodi [P] and Toledo–Tong [TT1]. What is more relevant to this paper is the Atiyah–Bott version of the Lefschetz fixed-point formula (see [AB]) adapted to cover the case where the set of fixed points is a submanifold. Such a version is due to Toledo and Tong (see [TT2]). The fixed-point formula we obtain in Section 2 for periodic self-maps is a little stronger than the classical formula when the characteristic of the ground field is positive: we get an identity in the Witt ring, which reduced modulo the characteristic yields the classical formula.

The Adams–Riemann–Roch is deduced from the fixed-point formula in Section 3. The Hirzebruch–Riemann–Roch is deduced from this in Section 4. The

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