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The Algebra of Jets

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To W. Fulton on the occasion of his 60th birthday

0. Introduction

0.1. Let A be a not necessarily commutative algebra over a commutative ring k, and let $d\colon A \to M$ be a derivation into an A-A-module. In this article we construct noncommutative k-algebras $J^0 = A, J^1, \dots,$ depending on d, that are A-A-modules, with natural inclusions $M \to J^n$ for $n \ge 1$ and with natural k-algebra homomorphisms $r^n\colon J^n \to J^{n-1}$. The algebras J^n fit into exact sequences $M^{\otimes_A n} \xrightarrow{i^n} J^n \xrightarrow{r^n} J^{n-1} \to 0$, where i^n is induced by the multiplication on J^n; the sequence is short exact when M is flat over A.

When M is free we show that J^n has a second natural structure as an algebra under a multiplication, the *shuffle* product. Denote J^n with this product by J^n_{shuff}. We show that J^n_{shuff} has a subalgebra J^n_{sym} of symmetric elements, depending on the choice of basis of M, and we have natural exact sequences $0 \to (M^{\otimes_A n})^{S_n} \xrightarrow{i^n} J^n_{\text{sym}} \xrightarrow{r^n} J^{n-1}_{\text{sym}} \to 0$.

When A is commutative we show that J^n always has a shuffle product, which coincides with the aforementioned product when M is free. In the commutative case we define, for every k-linear map $\varphi\colon M \to \bigwedge^2 M$ such that $\varphi d = 0$, a subalgebra J^n_φ of J^n_{shuff}; we also give natural conditions for J^n_φ to be equal to J^n_{sym} when M is a free A-module.

Finally, we show that the theory globalizes.

0.2. Let X be a Riemann surface. For each integer $n \ge 0$ there is a natural locally free $\mathcal{O}_X$-module $\mathcal{J}^n_X$ of rank $n+1$, called the *bundle of n-jets*, with a natural $\mathbb{C}$-linear map

$$\delta\colon \mathcal{O}_X \to \mathcal{J}^n_X.$$

The bundle, with the map δ, is characterized by the following property: Associated with each parameter x on an open subset U of X is an $\mathcal{O}_U$-basis $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$ of $\mathcal{J}^n_X\big|_U$, in terms of which $\delta\big|_U$ is given by the *Taylor expansion*,

$$\delta(f) = f\varepsilon_0 + (df/dx)\varepsilon_1 + \cdots + (d^n f/dx^n)\varepsilon_n. \tag{0.2.1}$$

More classically, the coefficients are taken with denominators $(1/i!)(d^i f/dx^i)$.

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