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On the WDVV Equation in Quantum K-Theory

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To W. Fulton on his 60th birthday

0. Introduction

Quantum cohomology theory can be described in general terms as intersection theory in spaces of holomorphic curves in a given Kähler or almost Kähler manifold X. By quantum K-theory we may similarly understand the study of complex vector bundles over the spaces of holomorphic curves in X. In these notes, we will introduce a K-theoretic version of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation which expresses the associativity constraint of the "quantum multiplication" operation on $K^*(X)$.

Intersection indices of cohomology theory,

$$\int_{[\text{space of curves}]} \omega_1 \wedge \cdots \wedge \omega_k$$

obtained by evaluation on the fundamental cycle of cup products of cohomology classes are to be replaced in K-theory by Euler characteristics

$$\chi(\text{space of curves}; V_1 \otimes \cdots \otimes V_k)$$

of tensor products of vector bundles. The hypotheses needed in the definitions of the intersection indices and Euler characteristics—that the spaces of curves are compact and nonsingular, or that the bundles are holomorphic—are rarely satisfied. We handle this foundational problem by restricting ourselves throughout the notes to the setting where the problem disappears. Namely, we will deal with the so-called moduli spaces $X_{n,d}$ of degree-d genus-0 stable maps to X with n marked points *assuming that X is a homogeneous Kähler space.* Under this hypothesis, the moduli spaces $X_{n,d}$ (we will review their definition and properties when needed) are known to be compact complex orbifolds (see [1; 10]). We use their fundamental cycle $[X_{n,d}]$, well-defined over $\mathbb{Q}$, in the definition of intersection indices, and we use sheaf cohomology in the definition of the Euler characteristic of a holomorphic *orbi-bundle* V:

$$\chi(X_{n,d}; V) := \sum (-1)^k \dim H^k(X_{n,d}; \mathcal{O}(V)).$$

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