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Good Representations and Solvable Groups

Dan Edidin & William Graham

Dedicated to William Fulton on his 60th birthday

1. Introduction

The purpose of this paper is to provide a characterization of solvable linear algebraic groups in terms of a geometric property of representations. Representations with a related property played an important role in the proof of the equivariant Riemann–Roch theorem [EG2]. In that paper, we constructed representations with that property (which we call freely good) for the group of upper triangular matrices in GL_n. We noted that it seemed unlikely that such representations exist for arbitrary groups; the main result of this paper implies that they do not.

To state our results, we need some definitions. A representation V of a linear algebraic group G is said to be *good* (resp. *freely good*) if there exists a nonempty G-invariant open subset $U \subset V$ such that

(i) G acts properly (resp. freely) on U.
(ii) $V \setminus U$ is the union of a finite number of G-invariant linear subspaces.

Note that freely good representations were called “good” in [EG2].

The main result of the paper is the following theorem.

Theorem 1.1. *Let G be a connected algebraic group over a field k of characteristic not equal to* 2. *Then G is solvable if and only if G has a good representation. Moreover, if G is solvable and k is perfect then G has a freely good representation.*

In characteristic 2, a solvable group still has good representations, and a partial converse holds (Corollary 4.1). A key step in the proof of the main result is Theorem 4.1, which is inspired by an example of Mumford [MFK, Ex. 0.4].

In characteristic 0, solvable groups are characterized by a weaker property that does not require the action to be proper. (In general, if G acts properly on X then G acts with finite stabilizers on X, but the converse need not hold.)

Theorem 1.2. *Let G be a connected algebraic group over a field of characteristic* 0. *Suppose that G has a representation V that contains a nonempty open set U such that*:

(1) *the complement of U is a finite union of invariant linear subspaces*; *and*
(2) *G acts with finite stabilizers on U.*

Then G is solvable.

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