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Polar Cremona Transformations

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Let $F(x_0, \dots, x_n)$ be a complex homogeneous polynomial of degree d. Consider the linear system $\mathcal{P}_F$ generated by the partials $\frac{\partial F}{\partial x_i}$; we call it the *polar linear system* associated to F. The problem is to describe those F for which the polar linear system is homaloidal, that is, for which the map $(t_0, \dots, t_n) \to \left(\frac{\partial F}{\partial x_0}(t), \dots, \frac{\partial F}{\partial x_n}(t)\right)$ is a birational map. We shall call F with such property a *homaloidal polynomial*. In this paper we review some known results about homaloidal polynomials and also classify them in the cases when F has no multiple factors and either $n = 3$ or $n = 4$ and F is the product of linear polynomials.

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1. Examples

As was probably first noticed by Ein and Shepherd-Barron [ES], many examples of homaloidal polynomials arise from the theory of prehomogeneous vector spaces. Recall that a complex vector space V is called *prehomogeneous* with respect to a linear rational representation of an algebraic group G in V if there exists a nonconstant polynomial F such that the complement of its set of zeros is homogeneous with respect to G. The polynomial F is necessarily homogeneous and an eigenvector for G with some character $\chi : G \to \mathrm{GL}(1)$, and it generates the algebra of invariants for the group $G_0 = \mathrm{Ker}(\chi)$. The reduced part F_{red} of F (i.e., the product of irreducible factors of F) is determined uniquely up to a scalar multiple. A prehomogeneous space is called *regular* if the determinant of the Hessian matrix of F is not identically zero; this definition does not depend on the choice of F. We shall call F a *relative invariant* of V. Note that there is a complete classification of regular irreducible prehomogeneous spaces with respect to a reductive group G (see [KS]).

THEOREM 1 [EKP; ES]. *Let V be a regular prehomogeneous vector space. Then its relative invariant is a homaloidal polynomial.*

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