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Linear Orbits of Arbitrary Plane Curves

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Dedicated to William Fulton on the occasion of his 60th birthday

0. Introduction

The Gromov–Witten invariants of $\mathbb{P}^2$ compute, roughly speaking, the number of plane curves of given degree d and genus g containing the appropriate number of general points. In recent years it has been discovered that these invariants are coherently linked together by the apparatus of quantum cohomology, which exposes their structure as d and g are allowed to vary.

For *nonsingular* plane curves, however, these invariants do not carry much information: the set of nonsingular curves of a given degree d is an open set of a projective space $\mathbb{P}^{d(d+3)/2}$, so the corresponding invariant is simply 1. We can consider a more refined question by fixing, as well as the degree d (and hence the genus $g = \frac{(d-1)(d-2)}{2}$), the moduli class in $\mathcal{M}_g$ of the curve. What data determines then the corresponding invariant? Can this invariant be effectively computed? Can other enumerative invariants be computed for the set of nonsingular curves of given degree and moduli class, such as the number of curves tangent to the appropriate number of general lines?

In this paper we fully answer these questions as well as a natural generalization of these questions to *arbitrary* (i.e., possibly singular, reducible, nonreduced) plane curves of any degree. The group PGL(3) of projective linear transformations of $\mathbb{P}^2$ acts naturally on the space $\mathbb{P}^{d(d+3)/2}$ parameterizing plane curves of degree d. Our main result is the computation of the degree of the closure in this space of the orbit of an arbitrary plane curve (in characteristic 0). Somewhat surprisingly, the enumerative geometers and the invariant theorists of the nineteenth century do not seem to have worked on this question. The orbit closure of a curve is a natural object of study, and its degree has a simple enumerative meaning: for a reduced curve with finite stabilizer, it counts the number of translates of the curve that contain eight given general points. For a nonsingular curve, this is the invariant just mentioned. In this sense, therefore, this problem is an isotrivial version of the problem of computing Gromov–Witten invariants.

The computation in this paper relies on our previous work on the subject, where we have dealt with special curves: nonsingular curves were treated in [AF2]; plane curves whose orbit has dimension less than $\dim \mathrm{PGL}(3) = 8$ are classified and

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