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McMillan's Area Problem

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1. Introduction

Let A denote the set of ideal accessible boundary points of a simply connected domain Ω. Recall that these are the finite radial limit points of the Riemann map from the unit disk onto Ω and that each radius along which the limit exists gives a distinct ideal boundary point. In particular, distinct ideal accessible boundary points may have the same complex coordinate. Fix $w_0 \in \Omega$ and for each $a \in A$ and $r < |w_0 - a|$ let $\gamma(a,r) \subset \{z : |z-a| = r\}$ be the circular crosscut of Ω separating a from w_0 that can be joined to a by a Jordan arc contained in $\Omega \cap \{z : |z-a| < r\}$. Throughout this paper we will refer to $\gamma(a,r)$ as the *principal separating arc* for a of radius r.

Let $L(a,r)$ denote the Euclidean length of $\gamma(a,r)$ and let

$$A(a,r) = \int_0^r L(a,\rho)\,d\rho.$$

In [5], McMillan showed that

$$\limsup_{r\to 0} \frac{A(a,r)}{\pi r^2} \geq \frac{1}{2}$$

almost everywhere on $\partial\Omega$ with respect to harmonic measure (denoted hereafter by a.e.-ω).

The purpose of this paper is to prove Theorem A.

Theorem A.

$$\liminf_{r\to 0} \frac{A(a,r)}{\pi r^2} \leq \frac{1}{2} \quad \textit{a.e.-}\omega.$$

This answers a question raised at the end of [5]. In an earlier paper [7], we proved the following theorem.

Theorem B.

$$\liminf_{r\to 0} \frac{L(a,r)}{2\pi r} \leq \frac{1}{2} \quad \textit{a.e.-}\omega.$$

This is also in answer to the last paragraph of [5]. Theorem A implies Theorem B but the basic idea of the proof is the same as in [7]. Let

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