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Level Sets and the Distribution of Zeros of Entire Functions

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1. Introduction

Let $S(A)$ denote the closed strip of width $2A$ in the complex plane $\mathbb{C}$ symmetric about the real axis:

$$S(A) = \{z \in \mathbb{C} : |\mathrm{Im}(z)| \le A\}, \tag{1.1}$$

where $A \ge 0$.

Definition 1.1. Let A be such that $0 \le A < \infty$. We say that a real entire function f belongs to the class $\mathfrak{S}(A)$ if f is of the form

$$f(z) = ce^{-az^2+bz}z^m \prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{z/z_k}, \tag{1.2}$$

where $a \ge 0$, m is a nonnegative integer, c is a nonzero real number, $z_k \in S(A) \setminus \{0\}$, and $\sum_{k=1}^{\infty} 1/|z_k|^2 < \infty$.

We allow functions in $\mathfrak{S}(A)$ to have only finitely many zeros by letting, as usual, $z_k = \infty$ and $0 = 1/z_k$ $(k \ge k_0)$, so that the canonical product in (1.2) is a finite product. The significance of the class $\mathfrak{S}(A)$ in the theory of entire functions is natural, since $f \in \mathfrak{S}(A)$ if and only if f is the uniform limit on compact sets of a sequence of real polynomials having zeros only in the strip $S(A)$ [dB, p. 202]. The Gauss–Lucas theorem [M, p. 22] tells us that this class of polynomials is closed under differentiation, and thus so is $\mathfrak{S}(A)$. The class $\mathfrak{S}(0)$ is also called the *Laguerre–Pólya class*, written $\mathcal{L}\text{-}\mathcal{P}$, so a function $f \in \mathcal{L}\text{-}\mathcal{P}$ has only real zeros.

For $f \in \mathfrak{S}(A)$, $\theta \in \mathbb{R}$, and $t \in \mathbb{R}$, we are interested in describing the distribution of zeros of the function

$$g_{t,\theta}(z) = e^{i\theta}f(z+t) + e^{-i\theta}f(z-t). \tag{1.3}$$

To motivate our results and to provide some background information, we recall from the literature (see e.g. [dB]) that if $f \in \mathfrak{S}(A)$ then the zeros of the entire function $g_{it,0}(z) = f(z+it) + f(z-it)$ $(t \in \mathbb{R})$ are "closer" to the real axis than those of f. More precisely, de Bruijn [dB, Thm. 8] showed that the zeros of $g_{it,0}$ all lie in the strip $S\big(\sqrt{A^2-t^2}\big)$ if $|t| < A$ and that $g_{it,0} \in \mathfrak{S}(0)$ if $0 \le A \le |t|$. The relationship between the zero set of $g_{it,0}$ and that of f has been studied by

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