Michigan Math. J. 47 (2000)

Invariant Vector Bundles of Rank 2 on Hyperelliptic Curves

Chanchal Kumar

1. Introduction

In classical projective geometry, the Segre cubic 3-fold $\mathbf{\Sigma}$ has been extensively studied in Baker [1] and Coble [4]. It is the GIT quotient $(\mathbb{P}^1)^6//\mathrm{PGL}(2,\mathbb{C})$ of $(\mathbb{P}^1)^6$ by the diagonal action of $\mathrm{PGL}(2,\mathbb{C})$ for the natural linearization on the line bundle $\boxtimes_{i=1}^6 \mathcal{O}_{\mathbb{P}^1}(1)$. It has been shown in Baker [1] and Coble [4] that the Segre cubic 3-fold arises on considering the linear system of quadrics in $\mathbb{P}^3$ that pass through five points in general position. The variety $\mathbf{\Sigma}$ thus embedded in $\mathbb{P}^4$ as a cubic hypersurface is actually the blow-up of $\mathbb{P}^3$ at these points, but with the proper transform of all lines joining any two points blown down to the ten nodes of $\mathbf{\Sigma}$. A general point $\omega \in \mathbf{\Sigma}$ of the Segre cubic 3-fold can obviously be interpreted as a curve $C = C_\omega$ of genus $g = 2$ with level 2-structure. Indeed, Van der Geer [12] showed that the variety dual to $\mathbf{\Sigma}$, which is a quartic 3-fold, can be identified with the Satake compactification of the moduli space $\mathcal{M}_{2,2}$ of smooth projective curves of genus $g = 2$ with level 2-structure.

A beautiful classical theorem (see [1; 4]) states that if $\omega \in \mathbf{\Sigma}$ is a general point then the *apparent contour*—namely, the locus of points of contact of tangent to $\mathbf{\Sigma}$ from this point ω—is the Kummer surface $\mathrm{Kum}(C)$ of the curve $C = C_\omega$ associated to $\omega \in \mathbf{\Sigma}$. In other words, the projection from the point ω maps $\mathbf{\Sigma}$ as a $2:1$ covering of $\mathbb{P}^3$ with Kummer surface $\mathrm{Kum}(C)$ as its branch locus and the apparent contour as its ramification locus. The composition of the birational map $\mathbb{P}^3 \dashrightarrow \mathbf{\Sigma}$ and the $2:1$ rational map $\mathbf{\Sigma} \dashrightarrow \mathbb{P}^3$ yields a $2:1$ rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, which is induced by the quadrics passing through six points in $\mathbb{P}^3$ in general position. The ramification locus of this rational map is called the *Weddle surface.* The Weddle surface with six nodes is a birational model of the Kummer surface. A nice modern account of these results may be found in the book by Dolgachev and Ortland [8].

The aim of this paper is to generalize all this beautiful geometry to higher dimensions. For $g \ge 2$, we consider the GIT quotient $(\mathbb{P}^1)^{2g+2}//G$ of $(\mathbb{P}^1)^{2g+2}$ by the diagonal action of $G = \mathrm{PGL}(2,\mathbb{C})$ for the natural G-linearization on the line bundle $\mathcal{L} = \boxtimes_{i=1}^{2g+2} \mathcal{O}_{\mathbb{P}^1}(1)$; we call it a *generalized Segre variety* or the *Segre g-variety* $\mathbf{\Sigma}_g$. We show that the Segre g-variety $\mathbf{\Sigma}_g$ is obtained by the linear system Ω of g-forms on $\mathbb{P}^{2g-1}$ that vanish with multiplicity $g-1$ through $2g+1$

Received May 1, 2000. Revision received August 23, 2000.