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Gehring's Lemma for Nondoubling Measures

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1. Introduction

Let $Q_0 \subset R^n$ be a fixed cube with sides parallel to the coordinate axes, let w be a strictly positive integrable function on Q_0, and let $1 < p < \infty$. We shall say that a positive function $g \in L^p_w(Q_0)$ belongs to $\mathrm{RH}_p(w)$ (i.e., that g satisfies a *reverse Hölder inequality*) if there exists a $C \ge 1$ such that, for every cube $Q \subset Q_0$ with sides parallel to the coordinate axes, we have

$$\left(\frac{1}{w(Q)}\int_Q g(x)^p w(x)\,dx\right)^{1/p} \le \frac{C}{w(Q)}\int_Q g(x)w(x)\,dx,$$

with $w(Q) = \int_Q w(x)\,dx$. If the underlying measure $\mu := w(x)\,dx$ satisfies the doubling condition—that is, if there exists a constant $c > 0$ such that $\mu(B(x,2r)) \le c\mu(B(x,r))$—then by Gehring's lemma [7] there exists an $\varepsilon > 0$ such that $g \in \mathrm{RH}_{p+\varepsilon}(w)$. For excellent accounts of the role that reverse Hölder inequalities play in PDEs, we refer to [9] and [11].

Recently there has been interest in extending the Calderón–Zygmund program to the context of nondoubling measures (cf. [1; 13; 14; 16; 20; 21] and the references therein). The purpose of this note is to prove Gehring's lemma for nondoubling measures of the form $\mu := w(x)\,dx$. Our main results are given in the next two theorems; for proofs, see Section 4. (When preparing the final version of this paper for publication we realized that Theorem 1 can be also obtained by a different method by means of combining Lemma 2.3 and Corollary 2.4 of [16] with Exercise 6.6 of [18].)

THEOREM 1. *Let $1 < p < \infty$, and let w be a positive integrable function on Q_0. Suppose that $g \in \mathrm{RH}_p(w)$. Then there exists an $\varepsilon > 0$ such that $g \in \mathrm{RH}_{p+\varepsilon}(w)$.*

THEOREM 2 (see [13] for the corresponding R^n version of this result; see [9] and the references therein for the doubling case). *Let g, h be positive functions in $L^p_w(Q_0)$ and suppose that there exists $c > 1$ such that, for all cubes $Q \subset Q_0$ with sides parallel to the coordinate axes, we have*

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